Review VI(Slides 333 - 365)

Vector Space & Sequence of Functions

HamHam

University of Michigan-Shanghai Jiao Tong University Joint Institute

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VV186 - Honors Mathmatics II

Vector Space

By introducing vector space, one can treat a specific group of functions which all have some shared properties to form such a set, then we can find out some convenient operation that will make us easier to deal with them.

We then make a specific definition to which **set** can be called a **Vector Space**.

Vector Space

We have eight axioms of vector space V (in $\mathbb{C}$ or $\mathbb{R}$)

$$+ : V \times V \to V$$

- i $(u+v)+w = u+(v+w)$
- ii $u+v = v+u$
- iii $\exists e \in V$ such that $v+e = e+v = v$
- iv $\underset{v \in V}{\forall} \underset{(-v) \in V}{\exists}$ such that $v+(-v) = (-v)+v = e$

$$\cdot : \mathbb{F} \times V \to V$$

- i $1 \cdot u = u \cdot 1 = u$
- ii $\lambda \cdot (u+v) = \lambda \cdot u + \lambda \cdot v$
- iii $(\lambda + \mu) \cdot u = \lambda \cdot u + \mu \cdot u$
- iv $(\lambda\mu) \cdot u = \lambda(\mu \cdot u)$

Vector Space

Common Misunderstanding

A real vector space is a subset of $\mathbb{R}^n$; a complex vector space is a subset of $\mathbb{C}^n$.

When we say a vector space is real, or complex, we just refer to the scalar multiplication – the scalar is real, or complex. We don't set extra limitation on the element of the vector space.

3. The set $\mathbb{C}^n$ is a real vector space if we define addition as in (3.3.1) and scalar multiplication

$$\lambda z := (\lambda z_1, \ldots, \lambda z_n), \qquad \lambda \in \mathbb{R},\ z \in \mathbb{C}^n.$$

Subspace

Let $(V, +, \cdot)$ be a real (complex) vector space and $U \subset V$. If $u_1 + u_2 \in U$ for $u_1, u_2 \in U$, and $\lambda u \in U$ for all $u \in U$ and $\lambda \in \mathbb{F}$, then $(U, +, \cdot)$ is a subspace of $(V, +, \cdot)$.

- This lemma actually states that, when the maps "+" and "·" makes sense in the subset U, then U will inherit the eight axioms of V.
- for a subspace, we don't need to check the 8 axioms as they are inherited from the original vector space. Only check the addition and product is closed is enough.
- A subset in a vector space that is not a subspace of it can also be a vector space. Why?

Recap: Metric

The definition of metric is as follows.

- $\forall x, y \in M,\ \rho(x, y) \geq 0$ and $\rho(x, y) = 0$ if and only if $x = y$.
- $\forall x, y \in M,\ \rho(x, y) = \rho(y, x)$.
- $\forall x, y, z \in M,\ \rho(x, z) \leq \rho(x, y) + \rho(y, z)$

Since the metric measure the distance between two points(elements) in the set, now we want to directly measure the length of one single element.

We modified our length function as follows.

- Still positive, and equal to zero if and only if this vector is 0 (e).
- Triangle inequality still holds(with some modification).
- The symmetric property makes no sense in this case, so...

"Metric" in Vector Space

The symmetric property makes no sense in this case, so...

Replace it by another important property in vector space. As you might notice, we care about scalar multiplication, so the new property is

The length of αu?

Equal to α times the length of u ,where u is a vector and α is a scalar.

Just as metric, we would like to give this function a name since it is so important, we will call it a **norm**. We then give the explicit definition of a norm.

Norm

Let V be a real(complex) vector space. Then a map

$$||\cdot|| : V \to \mathbb{R}$$

is called a norm if for all $u, v \in V$ and all $\lambda \in \mathbb{F}$, we have the following:

- $||\cdot|| \geq 0$ for all $v \in V$ and $||v|| = 0$ if and only if $v = 0$
- $||\lambda \cdot v|| = |\lambda| \cdot ||v||$
- $||u + v|| \leq ||u|| + ||v||$

Comment. Obviously, any normed can be considered as a metric of that space. However, not all the distance function generating metrics can be considered as a norm. A counterexample is:$\rho(x, y) = 0$ if $x = y$ and $\rho(x, y) = 1$ if $x \neq y$. The reason is when defining metric we don't assume the second property.

Norm

Examples

- $V = \mathbb{R}^n$, $||(a_1, a_2, \ldots, a_n)|| = \sum_{k=1}^{n} |a_k|$
- $V = C[a, b]$, $||f|| = \max_{x \in [a,b]} |f(x)|$
- $f : U \to V$, $||f|| = \sup_{x \in U \setminus \{0\}} \frac{||f(x)||_2}{||x||_1}$, where $|| \cdot ||_1$ is a norm defined on U and $|| \cdot ||_2$ is a norm defined on V.
- $V = C[a, b]$, $||u|| = \sup\{|u(z)| \cdot p(z) : z \in [a, b]\}$, where p is a real-valued function on $[a, b]$ and $0 < \alpha \leq p(z) \leq \beta$ for some $\alpha, \beta > 0$.

Comment. The third example is often called "operator norm"; while the last example is often called "weighted norm", a modification of which is useful in complex analysis.

More Examples in the Slides

3.3.9. Examples.

1. $\mathbb{R}^n$ with $\|x\|_2 = \left(\sum_{j=1}^{n} x_i^2\right)^{1/2}$,
2. $\mathbb{R}^n$ with $\|x\|_p = \left(\sum_{j=1}^{n} |x_i|^p\right)^{1/p}$ for any $p \in \mathbb{N} \setminus \{0\}$,
3. $\mathbb{R}^n$ with $\|x\|_\infty = \max_{1 \le k \le n} |x_k|$,
4. ℓ^∞ with $\|(a_n)\|_\infty = \sup_{n \in \mathbb{N}} |a_n|$,
5. c_0 with $\|(a_n)\|_\infty = \sup_{n \in \mathbb{N}} |a_n|$,
6. $C([a,b])$, $[a,b] \subset \mathbb{R}$, with $\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$.

Exercise

1. Prove the (reverse) triangle inequality for a norm

$$||\cdot|| : V \to \mathbb{R}.$$

That is, to prove

$$|\ ||x|| - ||y||\ | \le ||x \pm y||, \text{ where } x, y \in V$$

Exercise

2*. As you may have already seen, vector space is a wonderful way to classify things. Please show that the set of continuous functions is a vector space.

3*. Prove that a weighted norm is a norm on $C([a, b])$

Exercise

4. Check whether the following sentences are true or false:
   - Given a vector space V, and its two non-empty subspaces V_1, V_2, then $V_1 \cup V_2$ is a subspace of V.
   - Given a vector space V, and its two subspaces V_1, V_2, then $V_1 \cap V_2$ is a subspace of V.
   - The set of all linear maps on $\mathbb{R}$ is a subspace of $C(\mathbb{R})$.
   - Given a vector space $\mathbb{R}^n$, for any two distince norms $||\cdot||_1, ||\cdot||_2$ of $\mathbb{R}^n$, $||\cdot|| := \sqrt{||\cdot||_1 \cdot ||\cdot||_2}$ is also a norm of $\mathbb{R}^n$.
   - Given a vector space V, given two norms $||\cdot||_1 : V \to \mathbb{R}, ||\cdot||_2 : \mathbb{R} \to \mathbb{R}$, then the $||\cdot|| := ||\cdot||_2 \circ ||\cdot||_1$ is a norm of V.

Convergence in Vector Space

Now we have our length function in the vector space, namely a norm. Then we can talk about the convergence and continuity in vector space. We start with convergence.

Let $(V, ||\cdot||)$ be a normed vector space. A sequence in V is a map $(a_n) : \mathbb{N} \to V$. We say that (a_n) converges to $a \in V$ if

$$\underset{\varepsilon>0}{\forall} \ \underset{N\in\mathbb{N}}{\exists} \ \underset{n>N}{\forall} \ ||a_n - a|| < \varepsilon$$

Comment. We can use a similar method to define continuity. Besides, a more generalized concept of sequential convergence is: Let X be any non-empty set. A sequence in X is a map $(a_n) : \mathbb{N} \to X$. We say that (a_n) converges to $x \in X$ if there is some element of (a_n) in each neighborhood of x.

Continuity in Vector Space

Now let us introduce an interesting theorem.

Let $(V, ||\cdot||)$ be a normed vector space. The norm

$$||\cdot|| : V \to \mathbb{R}$$

is a continuous function on V.

Proof. By (reverse) triangle inequality,

$$|\ ||x|| - ||y||\ | \leq ||x - y||.$$

Fix arbitrary $\varepsilon > 0$, choose $\delta = \frac{1}{2}\varepsilon$ and we are done.

Exercise

6. Given a vector space $(V, ||\cdot||)$.

Let $a \in V$ be fixed; let $\lambda \neq 0 \in \mathbb{F}(\mathbb{C}$ or $\mathbb{R})$ be fixed. Prove the following.

- i The addition function $f: V \to V, f(x) = a + x$ is continuous and has a continuous inverse function.
- ii The scalar multiplication function $g: V \to V, g(x) = \lambda x$ is a continuous function and has a continuous inverse function.

Implement Vector Space in coding!

Hamster's terminal

文件(F) 编辑(E) 视图(V) 终端(T) 标签(A) 帮助(H)

```
niyinchen@hamster-virtualbox /media/sf_D_DRIVE/Files/VV186-TA/RC/RC6/code
 % make
g++ -Wall -c Vec.cpp
g++ -Wall -c main.cpp
g++ -Wall -o Vec Vec.o main.o
niyinchen@hamster-virtualbox /media/sf_D_DRIVE/Files/VV186-TA/RC/RC6/code
 % ./Vec
Vector a:   Vector: x = -1.2, y = -0.3
Vector b:   Vector: x = 3, y = 4
Vector a+b: Vector: x = 1.8, y = 3.7
Inner product of a and b: -4.8
Vector a times 10: Vector: x = -12, y = -3
niyinchen@hamster-virtualbox /media/sf_D_DRIVE/Files/VV186-TA/RC/RC6/code
 % |
```

Inner Product

Let $\mathbb{F}$ denote $\mathbb{R}$ or $\mathbb{C}$. An inner product in a real or complex vector space V is a map $(x, y) : V \times V \to \mathbb{F}$, such that the following holds:

- The inner product is linear in the first variable, i.e., for all $x, y, z \in V$ and all $\alpha, \beta \in \mathbb{F}$, $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$
- For all $x, y \in V$, $(x, y) = \overline{(y, x)}$ if V is complex, and $(x, y) = (y, x)$ if V is real.
- The inner product is positive definite, i.e., $(x, x) \geq 0$ for all x and $(x, x) = 0$ if and only if $x = 0$

If a vector space V is endowed with an inner product, we call it real (or complex) inner product space. If V is a real inner product space, then $(x, \alpha y + \beta z) = (\alpha y + \beta z, x) = \alpha(y, x) + \beta(z, x) = \alpha(x, y) + \beta(x, z)$

Examples

- $\mathbb{R}^n$ forms a real vector space with the inner product

$$(\cdot\ ,\ \cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},\ (x, y) = \sum_{i=1}^{n} x_i y_i$$

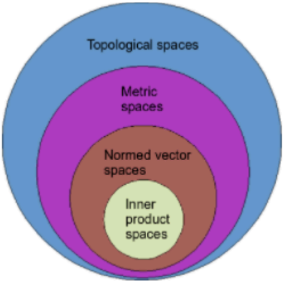
- $\mathbb{R}^2$ forms a real vector space with the inner product

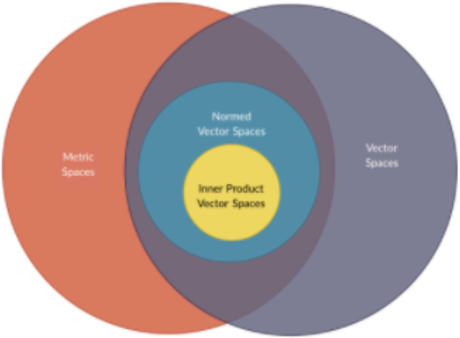
$$(\cdot\ ,\ \cdot) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R},\ (x, y) = 2x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2$$

- $\mathbb{C}$ forms a complex vector space with the inner product

$$(\cdot\ ,\ \cdot) : \mathbb{C} \times \mathbb{C} \to \mathbb{C},\ (x, y) = x\bar{y}$$

Diagram


Topological spaces
Metric spaces
Normed vector spaces
Inner product spaces


Metric Spaces
Normed Vector Spaces
Inner Product Vector Spaces
Vector Spaces

Convergence of Function Sequences

Let (f_n) be a sequence of real functions on $\Omega \subset \mathbb{R}$, then

1. Pointwise convergence. For every $x \in \Omega$,

$$f_n(x) \xrightarrow{(n\to\infty)} f(x) \quad :\Leftrightarrow \quad |f_n(x) - f(x)| \to 0$$

2. Uniform convergence.

$$f_n \xrightarrow{(n\to\infty)} f \quad :\Leftrightarrow \quad \sup_{x\in\Omega}|f_n(x) - f(x)| \to 0$$

Comment. For uniform convergence, we deal with the functions f_n as a whole, instead of each $f_n(x)$; for pointwise convergence, we deal with function values. Uniform convergence automatically implies pointwise convergence

Example

3.4.2. Example. The sequence (f_n),

$$f_n\colon [0,1] \to \mathbb{R}, \qquad f_n(x) = \begin{cases} 1 - nx, & 0 \le x \le 1/n, \\ 0, & \text{otherwise}, \end{cases}$$

converges to

$$f\colon [0,1] \to \mathbb{R}, \qquad f(x) = \begin{cases} 1, & x = 0, \\ 0, & \text{otherwise}, \end{cases}$$

pointwise, but not uniformly, as we now show.

How to find the limit

Since function vector space is abstract, we can use pointwise convergence to help us find the limit of a function sequence:

1. Calculate the pointwise limit f of a given function sequence (f_n).
2. Find a formula or estimate of $||f_n - f||$ for any $n \in \mathbb{N}$.
3. If $||f_n - f|| \to 0$ as $n \to \infty$, then (f_n) converges uniformly to f. Otherwise the convergence is not uniform.

Exercise

7. Calculate the limit of (f_n), sketch their graph, and determine whether the convergence is uniform or not.

- $f_n : \mathbb{R} \to \mathbb{R}, f_n(x) = \frac{(|x|^n)}{(1+|x|^n)}, n \in \mathbb{N}$
- $f_n : \mathbb{R} \to \mathbb{R}, f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}, n \in \mathbb{N}$
- $f_n : \mathbb{R} \to \mathbb{R}, f_n(x) = \frac{x^2+nx}{n}, n \in \mathbb{N}^*$

Exercises

8. Prove whether the following two statements are true or false
   - Every continuous function $f \in C([a, b])$ is the limit of a sequence of functions that is nowhere continuous
   - Every continuous function $g \in C([a, b])$ is the uniform limit of a sequence of nowhere differentiable continuous functions

Both of them are true!

Hint. For the second one, refer to slides 448.

Exercises

9. Let (f_n) be a sequence of functions in $C([a, b])$, and (f_n) converges to some function f uniformly. Prove that if $f \neq 0$ on $[a, b]$, then $(\frac{1}{f_n})$ converges to $\frac{1}{f}$ uniformly.

Exercises

10*.Let (f_n) be a sequence of functions such that for each $n \in \mathbb{N}$, $f_n \in C([a, b])$, $\forall x \in [a, b]$, $(f_n(x))$ is a bounded monotonic sequence.

- Please show that (f_n) converges point-wisely to some function f
- Suppose $f \in C([a, b])$, prove that (f_n) converges uniformly to f

L'Hopital's Rule

$$\lim_{x \searrow b} \frac{f(x)}{g(x)} = \lim_{x \searrow b} \frac{f'(x)}{g'(x)}, \text{ if } \lim_{x \searrow b} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \frac{\infty}{\infty} \text{ and } \lim_{x \searrow b} \frac{f'(x)}{g'(x)} \text{ exists.}$$

I am wrong last time

$$\lim_{x \to 1} \frac{x^3 + x - 2}{x^2 - 3x + 2} = \lim_{x \to 1} \frac{3x^2 + 1}{2x - 3} = \lim_{x \to 1} \frac{6x}{2} = 3$$

L'Hopital's Rule

Let f and g be real functions such that the interval $(C, \infty) \subseteq dom f \cap dom g$ and $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. Suppose further that f and g are defined and differentiable on (C, ∞) and $g'(x) \neq 0$ on it. Moreover, if the limit $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} =: L$ exists, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L$$

The only difference is that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$, but we still need to prove it.

Proof

Since $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L$, for any (fixed) $\varepsilon > 0$, there is some $M > C$ such that

$$\forall_{z>M} \left|\frac{f'(z)}{g'(z)} - L\right| < \varepsilon.$$

By the **Cauchy Mean Value Theorem**, for any $x > y > M$, there is some $z \in [y, x]$ such that

$$\left|\frac{f(x) - f(y)}{g(x) - g(y)} - L\right| = \left|\frac{f'(z)}{g'(z)} - L\right| < \varepsilon \text{ for } x > M$$

It follows that

$$\left|\frac{f(x)}{g(x)} - L\right| = \left|\frac{f(x) - f(y)}{g(x) - g(y)} \cdot \frac{f(x)}{f(x) - f(y)} \cdot \frac{g(x) - g(y)}{g(x)} - L\right| < 2\varepsilon$$

as x is large. Therefore, our proof is done.

RC Policy

- Always Pencil and Papers!
- Try to me more active. Feel free to interupt me at any time!

Instructions:

For the following question, the choice represent these sentences below.

(A) The content of RC classes are too easy and naïve, I feel it is a waste of time to go to his/her RCs.

(B) The content of RC classes are quite satisfying, but I hope he/she could do something more difficult/ cover more content.

(C) The TA is doing a good job in the RCs. I enjoy that and those content are tailored for me!

(D) The content of RC classes are quite satisfying, but I hope he/she could do something easier and explain things more detailly.

(E) The content of RC classes are too difficult and challenging, I feel he/she is going to kill me in his/her RCs.

(F) I never/seldom attend his/her RCs.

(c) Instructions

Attempts: 42 out of 43

What do you think for Ni Yinchen's RC (Tue.)?

(A)	3 respondents	7 %
(B)	6 respondents	14 %
(C)	13 respondents	30 %
(D)	2 respondents	5 %
(E)	1 respondents	2 %
(F)	17 respondents	40 %
No Answer	1 respondents	2 %

(d) Result

Received Feedbacks

- Hoping to see more difficult problems. I wonder if presenting a challenging problem at the end of the RC just for fun would be okay? Then maybe I can still get myself engaged when I have solved all the problems.
- Hope for releasing the RC slides in advance so that I can consider these questions before class, since time in class is so limited. Thanks!!!
- Provide solution for the homework.
- I hope my TAs can provide more sample exercise and explain more systematically how to complete different proves under different circumstance. More practice will help us have a deeper understanding of the proving methods and perform better in the exam.

Reference

- Exercises from 2019–Vv186 TA-Zhang Leyang.
- Exercises from 2020-Vv186 TA-Hu Pingbang.
- Exercises from 2020-Vv186 TA-Xia Yuxuan.

End

Thanks!