Review V(Slides 267 - 331)

Differentiation

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VV186 - Honors Mathmatics II

Differentiation – An Introduction

In order to investigate a function's derivative, we should first take a close look of **Linear map**.

Definition : A linear map on $\mathbb{R}$ is a function given by :

$$L : \mathbb{R} \to \mathbb{R}, \qquad L(x) = \alpha x, \alpha \in \mathbb{R}$$

Clearly, such a function has lots of good properties, which made our discussion becomes easier.

In this perspective, we would like to approximate any functions which we are interested in by a linear map. And if such linear map exists, we say this function is differentiable.

Differentiation – An Introduction

Translating into mathematical language...

Definition : Let $\Omega \subseteq \mathbb{R}$ be a set and $x \in \text{int}\Omega$. Moreover, Let $f : \Omega \to \mathbb{R}$ be a real function. Then we say f is **differentiable** if there exists a linear map L_x such that for all sufficiently small $h \in \mathbb{R}$,

$$f(x + h) = f(x) + L_x(h) + o(h) \quad \text{as } h \to 0$$

This linear map is **unique**, if it exists.

We call L_x "the derivative of f at x". If f is differentiable at all points of some open set $U \subseteq \Omega$, we say f is differentiable on U.

Derivative

Common misunderstandings:

$$L_x \text{ is a number for a fixed } x \in \Omega, \text{ because } L_x = \alpha.$$

L_x is **not a number**, but a **linear map**, or one can say "linear function", so it essentially is a function. $L_x \cdot h = \alpha \cdot h$ (for some α) doesn't mean $L_x = \alpha$.

To see this, one can consider a function given by

$$f(x) = 2x$$

,which doesn't mean $f = 2$.

Linear Map

- A more general case.

Derivative

Common misunderstandings:

$$\text{For } f(x) = x^4,\ f'(x) = 4x^3, \text{ so } L_x \text{ may not be linear}$$

You are confusing "derivative at a point" with "function that gives derivative". At certain point x, $4x^3$ is just a number in $\mathbb{R}$. Using our notation for L_x(or $f'(x)$), we can express L_x as

$$L_x(\cdot) = 4x^3(\cdot)$$

, the variable of L_x is not x, so L_x is **linear** for its input $(\cdot)$

Given a differentiable function $f : \Omega \to \mathbb{R}$, the function that gives a derivative can be denoted by $L : (\Omega \to \mathbb{R}) \to (\Omega \to \mathbb{R}),\ L(\cdot)(x) = L_x(\cdot)$. **It is a function that maps function to function.**

Derivative

Common misunderstandings:

The derivative of f at x is a line passing through $(x, f(x))$

Although it is usually a good idea to sketch something to help you to understand some mathematical concepts, but you always need to aware of the essential reason why such a graph make sense.

The derivative of f at x is a function, not a graph. We simply use the graph to illustrate our function sometimes, in this case($\mathbb{R}$), it will be a straight line, but in other case, it can be more complicated.

Rules of Differentiation

We now assume both f and g are differentiable functions, then:

- $(f+g)'(x) = f'(x) + g'(x)$
- $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$
- $(f \circ g)'(x) = f'(g(x))g'(x)$
- $(\frac{f}{g})'(x) = \frac{f'(xg(x) - f(x)g'(x))}{g^2(x)}$

Exercise

1. Practical calculation is really important! Please calculate the derivatives of the following functions.

- $(2x + 5x^2)^6$
- $\frac{\sqrt{x}}{x+1}$
- $\sqrt[3]{\frac{3x^2+1}{x^2+1}}$

2. More general Cases! Please calculate following functions' derivative. (Suppose g' always exists and doesn't vanish)

i. $f(x) = g(x \cdot g(a))$

ii. $f(x) = g(x + g(x)) + \frac{1}{g(x)}$

iii. $f(x) = g(x)(x - a)$

Inverse Function Theorem

Let I be an open interval and let $f : I \to \mathbb{R}$ be differentiable and strictly monotonic. Then the inverse map $f^{-1} : f(I) \to I$ exists and is differentiable at all points $y \in f(I)$ for which $f'(f^{-1}(y)) \neq 0$.

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

Demo

- Calculate $(\arctan x)'$

L'Hopital's Rule

$$\lim_{x\searrow b}\frac{f(x)}{g(x)}=\lim_{x\searrow b}\frac{f'(x)}{g'(x)}\text{ , if }\lim_{x\searrow b}\frac{f(x)}{g(x)}=\frac{0}{0}\text{ or }\frac{\infty}{\infty}\text{ and }\lim_{x\searrow b}\frac{f'(x)}{g'(x)}\text{ exists.}$$

What is wrong?

$$\lim_{x\to 1}\frac{x^3-x-2}{x^2-3x+2}=\lim_{x\to 1}\frac{3x^2-1}{2x-3}=\lim_{x\to 1}\frac{6x}{2}=3$$

Put a gun on your head: do write down the word "L'Hopital" !

Application of Differentiation

We list some useful Results and Theorems.

1. If a real function is differentiable at x, then it is continuous at x.
2. Hierarchy of local smoothness.
   1. Arbitrary function
   2. Function continuous at x
   3. Function differentiable at x
   4. Function continuously differentiable at x
   5. Function twice differentiable at x
   6. ...

Application of Differentiation

Result and Theorems.

3. Let f be a function and $(a, b) \subseteq \text{dom } f$ and open interval. If $x \in (a, b)$ is a maximum(or minimum) point of $f \subseteq (a, b)$ and if f is differentiable at x, then $f'(x) = 0$.

4. Let f be a function and $[a, b] \subseteq \text{dom } f$. Assume that f is differentiable on (a, b) and $f(a) = f(b)$. Then there is a number $x \in (a, b)$ such that $f'(x) = 0$.

   Comment. We need the requirement that f is **differentiable everywhere** on (a, b). Otherwise, a counterexample can be:

$$[a, b] = [0, 2], \quad \begin{cases} f(x) = x & x \in [0, 1] \\ f(x) = 2 - x & x \in (1, 2] \end{cases}$$

Application of Differentiation

Result and Theorems.

5. Let $[a, b] \subseteq \text{dom } f$ be a function that is continuous on $[a, b]$ and differentiable on (a, b). Then there exists a number $x \in (a, b)$ such that $f'(x) = \frac{f(b)-f(a)}{b-a}$.

6. Let f be a real function and $x \in \text{dom } f$ such that $f'(x) = 0$. If $f''(x) > 0$, then f has a local minimum at x, if $f''(x) < 0$, then f has a local maximum at x.

Comment

The case in which $f''(x) = 0$ is more complicated, different conditions may occur.

Example 1: $f'(x) = x^2$. Example 2: $f'(x) = x^3$.

As you can see from example 2, f may not even have a local extremum if $f''(x) = 0$.

Application of Differentiation

Result and Theorems.

7. Let f be a twice differentiable function on an open set $\Omega \subseteq \mathbb{R}$. If f has a local minimum at some point $a \in \Omega$, then $f''(a) \geq 0$.

Proof :

Suppose f has a local minimum at a. If $f''(a) < 0$, then f would also have a local maximum at a. Thus, f would be constant in some interval containing a. So $f''(a) = 0$. But this contradicts to our assumption.

Comment. An analogous statement is : If f has a local maximum at some point $a \in \Omega$, then $f''(a) \leq 0$.

Exercise

3. This exercise aims to show that differentiation can also be used to prove sequential results. Recall the inequality (see also review 2)

$$|a+b|^n \leq 2^{n-1}(|a|^n + |b|^n)$$

Now try to use differentiable function to prove it.

Exercise

4. Prove that if

$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0$$

Then $a_0 + a_1 x + \cdots + a_n x^n = 0$ for some $x \in [0, 1]$

Exercise

5. Suppose that f satisfies $f'' + f'g - f = 0$ for some function g. Prove that if f is 0 at two distinct points, then f is 0 on the interval between them.

Convexity and Concavity

For further analysis of functions, we would introduce the concept of **Convexity** and **Concavity**.
The definition of these two concepts are as follows.

Let $\Omega \subseteq \mathbb{R}$ be any set and $I \subseteq \Omega$ an interval. A function $f : \Omega \to \mathbb{R}$ is called convex on I if for all

$$x, a, b \in I \text{ with } a < x < b, \frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a}$$

A strictly convex function is a function that satisfies

$$\frac{f(x) - f(a)}{x - a} < \frac{f(b) - f(a)}{b - a}. \tag{1}$$

We say a function f is concave if $-f$ is convex. We say a function f is strictly concave if $-f$ is strictly convex.

Convexity and Concavity

Comment 1.

We often use ”−”(minus sign) to define a new definition from an existing one. The benefit is that these two definitions can be strongly related with each other.

Convexity and Concavity

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Comment 2.

There is a quick way to memorize it... Con**cave**...

Convexity and Concavity

Results/Theorem & Comment

1. Let $f\colon I \to \mathbb{R}$ be strictly convex on I and differentiable at $a, b \in I$. Then:

   i For any $h > 0 (h < 0)$ such that $a + h \in I$, the graph of f over the interval $(a, a + h)$ lies below the secant line through the points $(a, f(a))$ and $(a + h, f(a + h))$

   ii The graph of f over all I lies above the tangent line through the point $(a, f(a))$

   iii If $a < b$, then $f'(a) < f'(b)$

Draw some pictures to visualize these results!

Convexity and Concavity

Results/Theorem & Comment

2. A function $f : I \to \mathbb{R}$(I is an interval) is convex if and only if

$$\underset{t\in(0,1)}{\forall}\ \underset{x,y\in I}{\forall} \text{ with } x < y,\ f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

Draw some pictures to visualize these results!

3. Let I be an interval, $f : I \to \mathbb{R}$ differentiable and f' strictly increasing. If $a, b \in I$, $a < b$ and $f(a) = f(b)$, then

$$f(x) < f(a) = f(b) \text{ for all } x \in (a, b)$$

Exercise

6. This exercise will show why convexity is useful.

   i Let f be a convex function on $[a, b]$. Prove that

   $$f(\sum_{i=1}^{n}\lambda_i x_i) \leq \sum_{i=1}^{n}\lambda_i f(x_i),\ \ x_i \in [a, b],\ \ \sum_{i=1}^{n}\lambda_i = 1,\ \ \lambda_i > 0$$

   This inequality is known as **Jensen's Inequality**(for discrete measure.)

   ii Show that

   $$\prod_{i=1}^{n} a_i^{\lambda_i} \leq \sum_{i=1}^{n}\lambda_i a_i,\ \ a_i \geq 0,\ \ \sum_{i=1}^{n}\lambda_i = 1,\ \ \lambda_i > 0.$$

   This is the inequality you will encounter in your assignment.

Exercise

7^*. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Prove that if f satisfies

$$f(\frac{x_1 + x_2}{2}) \leq \frac{1}{2}(f(x_1) + f(x_2))$$

, where $x_1, x_2 \leq [0, 1]$, then f is convex.

Exercise

8*. Let f be a continuous convex real function on $[a, b]$. Show that f either has one local minimum or infinitely many local minimums on $[a, b]$.

Additional Exercise

9. Suppose $f\colon [0, n], n \in \mathbb{N}$ is a continuous function, and is differentiable on $(0, n)$. Furthermore, assume that

$$f(0) + f(1) + \cdots + f(n-1) = n,\ \ f(n) = 1$$

Show that there must exist $c \in (0, n)$ such that $f'(c) = 0$.

Additional Exercise

10*. In this exercise, we would like to give a deeper investigation of Lipschitz condition. If a real function $T : \Omega \to \mathbb{R}$ satisfies

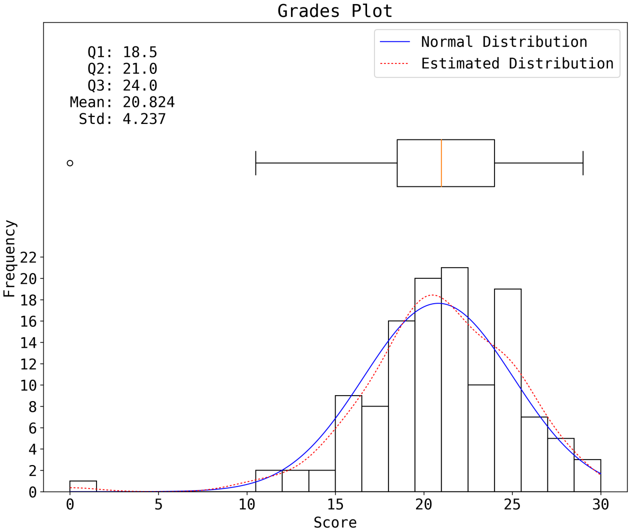
$$|T(x) - T(y)| \leq k \cdot |x - y|^{\alpha}$$

for any $x, y \in \Omega$, we say T satisfies "Lipschitz condition of order α".

1. Show that if $\alpha > 0$, then T is continuous.
2. Show that if $\alpha > 1$, then T is a constant function, i.e.,

$$\underset{C \in \mathbb{R}}{\exists} \; T(x) = C$$

About Mid I


Grades Plot
Normal Distribution
Estimated Distribution
Q1: 18.5
Q2: 21.0
Q3: 24.0
Mean: 20.824
Std: 4.237
Frequency
0
2
4
6
8
10
12
14
16
18
20
22
0
5
10
15
20
25
30
Score

Reference

- Exercises from 2019–Vv186 TA-Zhang Leyang.
- Exercises from 2020-Vv186 TA-Hu Pingbang.

End

Thanks!