Review IV(Slides 170 - 250)

Real Functions

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VV186 - Honors Mathmatics II

Common Types & Manipulating Functions

Common function types (these should be familiar from high school):

- Power functions: x^n
- Polynomials: Combinations of power functions
- Rational functions: Quotient of polynomials
- Piecewise functions: "sticking" functions together
- Periodic functions: $f(x+T) = f(x)$

Manipulating functions (also familiar from high school):

- $f(k \cdot x)$
- $k \cdot f(x)$
- $f(x+t)$
- $f(x)+t$

Addition, Multiplication and Composition

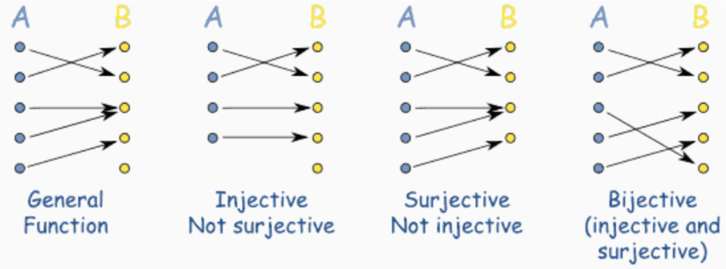
Let $f : X_1 \to Y_1, g : X_2 \to Y_2$, and let $E = X_1 \bigcap X_2$, then

$$f + g := f(x) + g(x), x \in E$$

$$f \cdot g := f(x) \cdot g(x), x \in E$$

$$f \circ g : X_2 \to Y_1, x \mapsto f(g(x)), \text{ if } g(x) \in Y_2 \cap X_1 \neq \emptyset$$

Some terms..



Try to remember it...perhaps?

Limit

Two Definitions:

(1) (Common) Let f be a real- or complex-valued function defined on a subset $\Omega \subset \mathbb{R}$ and let x_0 be an accumulation point of f. Then the limit of f as $x \to x_0$ is equal to $L \in \mathbb{C}$, written

$$\lim_{x \to x_0} f(x) = L :\Leftrightarrow \underset{\delta > 0}{\exists} \underset{x \in \Omega \setminus \{x_0\}}{\forall} |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

(2) (Sequential) Let f be a real- or complex-valued function defined on a subset $\Omega \subset \mathbb{R}$ and let x_0 be an accumulation point of f. Then the limit of f as $x \to x_0$ is equal to $\varepsilon\mathbb{C}$, written

$$\lim_{x \to x_0} f(x) = L :\Leftrightarrow \underset{a_n \subset \Omega}{\forall} a_n \to x_0 \Rightarrow f(a_n) \to L$$

Common Results

Let f, g be two real functions with the same domain $\Omega \subset \mathbb{R}$. Furthermore, suppose $\lim_{x \to x_0} f(x)$ and $\lim_{x \to x_0} g(x)$ exists at some point $x_0 \in \Omega$. Then:

- $\lim_{x \to x_0} [f(x) + g(x)] = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)$
- $\lim_{x \to x_0} [f(x) \cdot g(x)] = \lim_{x \to x_0} f(x) \cdot \lim_{x \to x_0} g(x)$
- $\lim_{x \to x_0} [f(x)/g(x)] = \lim_{x \to x_0} f(x) / \lim_{x \to x_0} g(x)$, if $\lim_{x \to x_0} g(x) \neq 0$

(You are encouraged to prove these results, the proof is similar for that of sequence)

Landau Symbols

My interpretation:

- Big-O : As Large As...

$$f(x) = O(\phi(x)), x \to \infty : \underset{C>0}{\exists} \underset{M>L}{\exists} x > M \Rightarrow |f(x)| \leq C|\phi(x)|$$

- Small-o : To Small so I don't care at all

$$f(x) = o(\phi(x)), x \to x_0 : \underset{C>0}{\forall} \underset{\varepsilon>0}{\exists} \underset{x\in\Omega\setminus\{x_0\}}{\forall} |x-x_0| < \varepsilon \Rightarrow |f(x)| < C|\phi(x)|$$

But still, definitions are important for proof!

Example

- Physics: $(1+x)^n \approx 1 + nx, x << 1$
- Time Complexity for bubble sort: $O(n^2)$

Landau Symbols

Some common results(see also in assignments):

- $O(g(x)) + O(f(x)) = O(|g(x)| + |f(x)|)$
- $O(f(x))(g(x)) = (f(x)g(x))$
- $O(f(x))o(g(x)) = o(f(x))(g(x))$
- $O(O(f(x))) = O(f(x))$
- $o(O(f(x))) = o(f(x))$

Comment:Some of the results will be useful when dealing with differentiation

Continuty

Let $\Omega \subset \mathbb{R}$ be any set and $f : \Omega \to \mathbb{R}$ be a function defined on Ω. Let $x \in \Omega$. We say that f is continuous at x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$.

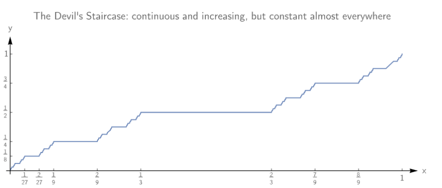
If f is continuous at all points $x \in U \subset \mathbb{R}$. Then we say f is continuous at **every point** of U, or simply, f is continuous on U.

Quick check:

- How to prove that a function is continuous?
- How to prove that a funtion is continuous at one point x_0?

Concepts for continuous extension and one-side continuty....

Something Strange...



(a) The Devil's Staircase

(b) The Riemann Function

Results/Theorems for Continuous Real Functions

(1) Let $\Omega \subset \mathbb{R}$ be some set and $f : \Omega \to \mathbb{R}$ be a function that is continuous at some point $x_0 \in \Omega$ and assume that $f(x_0) > 0$. Then there exists a $\delta > 0$ such that $f(x) > 0$ for all $x \in (x_0 - \delta, x_0 + \delta) \cap \Omega$. (Slide 227)

(2) Let $a < b$ and $f : [a, b]\mathbb{R}$ be a continuous function with $f(a) < 0 < f(b)$. Then there exists some $x \in [a, b]$ such that $f(x) = 0$. (Slide 228)

(3) Let $a < b$ and $f : [a, b]\mathbb{R}$ be a continuous function. Then for $y \in [\min\{f(a), f(b)\}, \max\{f(a), f(b)\}]$ there exists some $x \in [a, b]$ such that $y = f(x)$. (Slide 230)

Results/Theorems for Continuous Real Functions

(4) Let $f\colon [a, b] \to \mathbb{R}$ be a continuous function with $ran\ f \subset [a, b]$. Then f has a fixed point, i.e., there exists some $x \in [a, b]$ such that $f(x) = x$. (Slide 231)

(5) Let $\Omega \subset \mathbb{R}$ be some set and $\colon \Omega \to \mathbb{R}$ be a function that is continuous at some point $x_0 \in \Omega$. Then there exists a $\delta > 0$ such that f is bounded above on $(x_0 - \delta, x_0 + \delta) \cup \Omega$. (Slide 233)
Comment: This Lemma mainly deals with the behavior of f locally (on some neighborhood)

(6) Let $a < b$ and $f\colon [a, b] \to \mathbb{R}$ be a continuous function. Then f is bounded above. (Slide 234)

Results/Theorems for Continuous Real Functions

(7) Let $a < b$ and $f: [a, b] \to \infty$ be a continuous function. Then there exists a $y \in [a, b]$ such that $f(x) \le f(y)$ for all $x \in [a, b]$. (Slide 236)
Comment. This theorem ensures that f takes its maximum and minimum on $[a, b]$.

(8) Let $f: I \to \mathbb{R}$ be a continuous function, where I is an interval. Then f is strictly monotonic if and only if f is bijective. (Slide 240 & 244)

(9) Let $f: I \to \mathbb{R}$ be a continuous function. Suppose $[a, b] \subset I$, then $f([a, b])$ is an interval. (Slide 251)

They are important! Make sure you have time to digest these!

Uniform continuity

Interpretation: not driving to fast!
Example: $f(x) = \sin \frac{1}{x}$ on the interval (0,1) is not uniformly continuous

Uniform Continuty Theorem:
Let $f : I \to \mathbb{R}$ be a continuous function. Suppose $[a, b] \subset I$, then f is uniformly continuous on $[a, b]$. (Slides 249)

Exercises

1. Discuss where is the following function continuous, where is it not continuous. No proof needed.

(i) $f(x) = \frac{\sqrt{3}(1+4x)-1}{2\sin x}$

(ii) $f(x) = \lceil x \rceil, x > 0$

(Adapted from SJTU Math textbook, P69)

(It's necessary to being able to investigate concrete functions)

Exercises

2. Please prove, or disprove by giving counterexamples of the following statements:

(i) $(1+O(x))^2 = 1 + O(x^2)$ as $x \to 0$

(ii) $o(x)^n = o(x^n), n \in \mathbb{N}^*$, as $x \to 0$

What about $x \to \infty$? Would the result change?

Exercises

3. Let $f\colon [0,+\infty) \to \mathbb{R}$ be a continuous function such that $\lim_{x\to\infty} f(x)$ exists and is finite. Show that f is uniformly continuous on $[0,+\infty)$.

Exercises

4. Let $f : [a, b] \to \mathbb{R}$ be a continuous function with $b > a$. Given $\varepsilon > 0$, show that there is a polygonal function g such that $|f(x) - g(x)| < \varepsilon$ for all $x \in [a, b]$.

Note: A polygonal function is a function formed by a finite number of line segments. Of course, a polygonal function is continuous.

- Try to write a complete proof!

Exercises

5*. The function $f(x)$ is defined on interval I. Proof that $f(x)$ is uniformly continuous if and only if: for any sequence $x'_n, x''_n \subset I$, if $\lim_{n\to\infty}(x'_n - x''_n) = 0$, then $\lim_{n\to\infty}(f(x'_n) - f(x''_n)) = 0$.

Exercises

6*.Let $f\colon (0,+\infty) \to \mathbb{R}$ be a continuous function such that $f(x^2) = f(x)$. Please show that f is a constant function on $(0,+\infty)$, i.e.,

$$\exists M \in \mathbb{R}, \forall x \in domf, f(x) = M$$

(SJTU Math textbook, P70)

Exercises

7*. Suppose the right endpoint for an interval I_1 is $c \in I_1$, the left endpoint for an interval I_2 is also $c \in I_2$(I_1, I_2 can be infinite interval or just finite). Prove that if f is uniformly continuous on I_1 and I_2, then f is uniformly continuous on $I = I_1 \cup I_2$.

Exercises

8. Let $f : \Omega \to \mathbb{R}$ be a real function that satisfies Lipschitz condition, that is, there is a constant $M > 0$ such that for all x and y in the domain of f, $|f(x) - f(y)| \leq M|x - y|$.

(i) Show that f is uniformly continuous

(ii) Now Let $\Omega =: [a, +\infty)$, where $a > 0$. Show that $f(x)/x$ is uniformly continuous

Intergration Bee!

Horst and My TAs plan to hold 2-nd JI Intergration Bee this semester, probably in early November. The problems would include:

- single variable intergration(you'll learn this in vv186).
- multi variable intergration(e.g. surface intergal, learn in vv285)
- ordinary differential equations(learn in vv286).

If you're interested, feel free to contact us!

Midterm Reminder

- The first midterm exam is scheduled onto next Tuesday, lecture time.
- Big RC and OH will be held on Sunday (time will be announced later).
- Tips for reviewing:
  1. Go through the lecture slides. Recite the definitions and theorem statements. If time permits, review the proofs of theorems and lemmas.
  2. Go through the assignments. Pay attention to the definitions in them ($\overline{\lim}$, $\underline{\lim}$, etc.)
  3. Do the sample exams and check the answers.
  4. Attend the big RC unless you are very confident.
- Remember to sleep well the night before the exam!
- Good luck!

Reference

- Exercises from 2019–Vv186 TA-Zhang Leyang.
- Exercises from 2020-Vv186 TA-Hu Pingbang.
- SJTU Math Textbook.
- Mathmatical Analysis I. *Department of Mathematics, ECNU*,,4-th version. Beijing: Higher Education Press. 2016.3 print.