Exercises for Midterm 2

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VV186 - Honors Mathmatics II

Exercises

1. Suppose that $(V, +, \cdot)$ is a vector space and let $U, W \subset V$ be two subspaces. Then:

(A) $U \cap W \neq \emptyset$

(B) $V \setminus U$ is also a subspace of V

(C) $U \cup W$ is a subspace of V

(D) $U \cap W$ is a subspace of V

Exercises

2. Let $(a, b) \subset \mathbb{R}$ be an open interval and denote $C^1(a, b) \cup C([a, b])$ the vector space of those functions on (a, b) that are continuous on $[a, b]$ and continuously differentiable on (a, b). On this space a norm is defined by

(A) $||f|| := \sup_{x \in [a,b]} |f(x)|$

(B) $||f|| := \sup_{x \in [a,b]} |f'(x)|$

(C) $||f|| := \sup_{x \in [a,b]} |f(x)| + \sup_{x \in [a,b]} |f'(x)|$

(D) $||f|| := \sup_{x \in [a,b]} (|f(x)| + |f'(x)|)$

Exercises

3. Let (a_n) be a sequence of real numbers such that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges at least for $x \in [0, 1]$. Then $f(x)$

(A) $f(x)$ must converge for $x = -1/2$

(B) $f(x)$ must converge for $x = -1$

(C) $f(x)$ may or may not converge for $x = -1$

(D) $f(x)$ never converges for $x = 2$

Exercises

4. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is twice differentiable and has a local minimum at $x = 0$. Then

(A) f is convex in a neighborhood of $x = 0$

(B) $f''(0) > 0$

(C) $f''(0) \geq 0$ and $f''(0) = 0$ is possible

(D) $f'(x)$ is increasing in a neighborhood of $x = 0$

Exercises

5. Let V be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. If one were to define

$$U_1 + U_2 := \{z \in V : \underset{x \in U_1}{\exists} \underset{y \in U_2}{\exists} z = x + y\}$$

$$U_1 - U_2 := \{z \in V : \underset{x \in U_1}{\exists} \underset{y \in U_2}{\exists} z = x - y\}$$

for subspaces U_1, U_2 of V, then one would have (Adapted from Vv285 practice quiz)

(A) $U_1 - U_2 = 0$

(B) $(U_1 - U_2) + U_2 = U_1$

(C) $U_1 - U_2 = U_1 + U_2$

Exercises

6. (i) Does the following series converge?

$$\sum_{n=0}^{\infty} \frac{(2n)!(3n)!}{n!(4n)!}$$

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$$\sum_{n=0}^{\infty} \frac{(2n)!(3n)!}{n!(4n)!}$$

(ii) For which $a \in \mathbb{R}$ does the following series converge?

$$\sum_{n=0}^{\infty} \left(\frac{1}{n} - \sin\left(\frac{1}{n}\right)\right)^{a}$$

Exercises

7. Let $f\colon [0,2\pi] \to \mathbb{R}$ be given by

$$f(x) = \frac{1}{1 + e^{\pi - x}\sin(x)}$$

(i) For which x is $f'(x) = 0$? Derive the solution to the equation $\sin(x) = \cos(x)$, $x \in [0, 2\pi]$

(ii) Where is f increasing? Where is f decreasing?

(iii) Find the local extrema of f

(iv) What can you say about the convexity and concavity of f?

(v) Sketch the graph of f, clearly indicating any siginicant features of the graph

Exercises

8*. Suppose that (f_n) is a sequence of increasing functions $f_n : [0,1] \to [0,1]$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad \text{for all } x \in [0,1],$$

Suppose that f is a continuous function. Show that the convergence is uniform.

(Note that the functions f_n are not assumed to be continuous.)

Comment. This is also called *Dini's theorem.*

Exercises

9. Let (f_n) be a sequence of functions in $C([a,b])$, and (f_n) converges to some function f uniformly. Prove that if $f \neq 0$ on $[a,b]$, then $(\frac{1}{f_n})$ converges to $\frac{1}{f}$ uniformly.

Exercises

10. Prove that if a positive series a_n diverges, then

$$\sum \frac{a_n}{1+a_n}$$

also diverges.

End

Good Luck!

Reference

- Exercises from 2020 Vv186 Midterm 2.
- Exercises from 2021 Vv285 Practice Quiz
- Exercises from my RC4-RC7.