# Part 0-Math Foundation

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October 24, 2021

VV186 - Honors Mathmatics II

# About Exam

- 30 pts, possibly contain
  1. Multiple choice questions (The choices can be all wrong.)
  2. Calculation question
  3. Proof question
  4. Explanation question
  5. ...
- 100 mins (8:00 - 9:40). Do Wake Up!
- $100/30 = 3.3$ mins/pt. Therefore, you probably don't want to spend more than 5 mins on 1 pt.
- Don't panic if you cannot figure out all some specific question. Just skip it and do it later.
- The questions may not be arranged in the order of difficulty.

# Checking List

- Statement/Truth Table
- Prove by Contraposition
- Logical/Sets Operation
- Cartesian product/ Ordered pairs
- Properties of Natural/Rational/Complex Numbers
- Concept of Interval/Points/Sets
- **Mathematical Induction**

Part 0 is **not** the key point of the exam. However, you should still go through these concepts. Though they might not be directly tested, they could occur somewhere in your exam paper.

# Truth Table

**Exercise 1.2**

i) Let $a, b$ be statements. Write out the truth tables to prove *de Morgan's rules*:

$$\neg(a \wedge b) \equiv \neg a \vee \neg b, \qquad \neg(a \vee b) \equiv \neg a \wedge \neg b.$$

**(2 Marks)**

Notice that the last column of the truth table should be an evaluation on equivalence. For example, for the first question, you should evaluate $\neg(a \wedge b) \Leftrightarrow \neg a \vee \neg b$. It is supposed to be true for the whole column. Only through this fact can you conclude that $\neg(a \wedge b) \equiv \neg a \vee \neg b$.

Please distinguish equivalence ($\Leftrightarrow$) and logical equivalence ($\equiv$). The former one is a binary operation, while the later one indicates a relationship between two compound statements. To be more precise, logical equivalence indicates that the equivalence between two compound statements is a tautology.

# Contraposition

The famous **tautology**

$$(A \Rightarrow B) \equiv (\neg B \Rightarrow \neg A)$$

is useful in some situation. If you are asked to prove some statement $A \Rightarrow B$ but you cannot find a simple way, you can try to prove by contraposition.

# Exercise

Let $x \in \mathbb{Z}$. If $x^2 - 6x + 5$ is even, then $x$ is odd.

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Let $x \in \mathbb{Z}$. If $x^2 - 6x + 5$ is even, then $x$ is odd.

**Proof:** Suppose that $x$ is even. Then we want to show that $x^2 - 6x + 5$ is odd. Write $x = 2a$ for some $a \in \mathbb{Z}$, and plug in:

$$\begin{aligned} x^2 - 6x + 5 &= (2a)^2 - 6(2a) + 5 \\ &= 4a^2 - 12a + 5 \\ &= 2(2a^2 - 6a + 2) + 1 \end{aligned}$$

Thus $x^2 - 6x + 5$ is odd.

# Mathematical Induction

Often one wants to show that some statement frame $A(n)$ is true for all $n \in \mathbb{N}$ with $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Mathematical induction works by establishing two statements:

(I) $A(n_0)$ is true.

(II) $A(n+1)$ is true whenever $A(n)$ is true for $n \geq n_0$, i.e.,

$$\forall_{\substack{n \in \mathbb{N} \\ n \geq n_0}} A(n) \Rightarrow A(n+1)$$

**Remark:** Mathematical Induction is basically the only relatively important concept in Part 0 of VV186. Please see Ex5, Ex6 and Ex7 in sample exam and see the related rubric.

# Exercise

Let $a_n$ be the following expression with $n$ nested radicals:

$$a_n = \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + \sqrt{2}}}}$$

Prove that $a_n = 2\cos\frac{\pi}{2^{n+1}}$

## Exercise

**Proof:** Note that $a_n$ can be defined recursively like this: $a_1 = \sqrt{2}$, and $a_{n+1} = \sqrt{a_n + 2}$ for $n \geq 1$. We proceed by induction. For $n = 1$ we have in fact $a_1 = \sqrt{2}$, and $2 \cdot \cos\frac{\pi}{4} = 2 \cdot \frac{1}{\sqrt{2}} = \sqrt{2}$.

Next, assuming the result is true for some $n \geq 1$, we have

$$\begin{aligned} a_{n+1} = \sqrt{2 + a_n} &= \sqrt{2 + 2\cos\frac{\pi}{2^{n+1}}} \\ &= \sqrt{2 + 2\cos 2\frac{\pi}{2^{n+1}}} \\ &= \sqrt{2 + 2(2\cos^2\frac{\pi}{2^{n+2}} - 1)} \\ &= \sqrt{4\cos^2\frac{\pi}{2^{n+2}}} = 2\cos\frac{\pi}{2^{n+2}} \end{aligned}$$

By induction, we conclude that $a_n = 2\cos\frac{\pi}{2^{n+1}}$.

# Complex Numbers

Given $z_1 = (a_1, b_1)$ and $z_2 = (a_2, b_2)$

- $z_1 + z_2 = (a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1, b_2)$
- $z_1 \cdot z_2 = (a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2 - b_1 b_2, a_1 b_2 - a_2 b_1)$
- $c \cdot z_1 = c(a_1, b_2) = (ca_1, cb_1), c \in \mathbb{R}$
- $\bar{z_1} = (a_1, -b_1)$
- $|z_1|^2 = a_1^2 + b_1^2 = z_1 \bar{z_1}$
- $\text{Re } z_1 = \frac{z_1 + \bar{z_1}}{2}$
- $(\text{Im } z_1) i = \frac{z_1 - \bar{z_1}}{2}$
- $2(|z_1|^2 + |z_2|^2) = |z_1 + z_2|^2 + |z_1 - z_2|^2$
- $|z_1 + z_2| \leq |z_1| + |z_2|$ **(Triangle Inequality)**

Note that the ordering relation is not defined in $\mathbb{C}$!

# Concept of Interval/Points/Sets

- Interior/Exterior/Boundary point
- Accumalation point
- Open/Closed Set
- Open/Closed/Half-open interval
- Closure ($\overline{I} = \partial I \cup I$)
- bounded/unbounded
- max/min
- sup/inf
- lim sup/lim inf
- open ball ($B_\varepsilon(a)$)

# Example

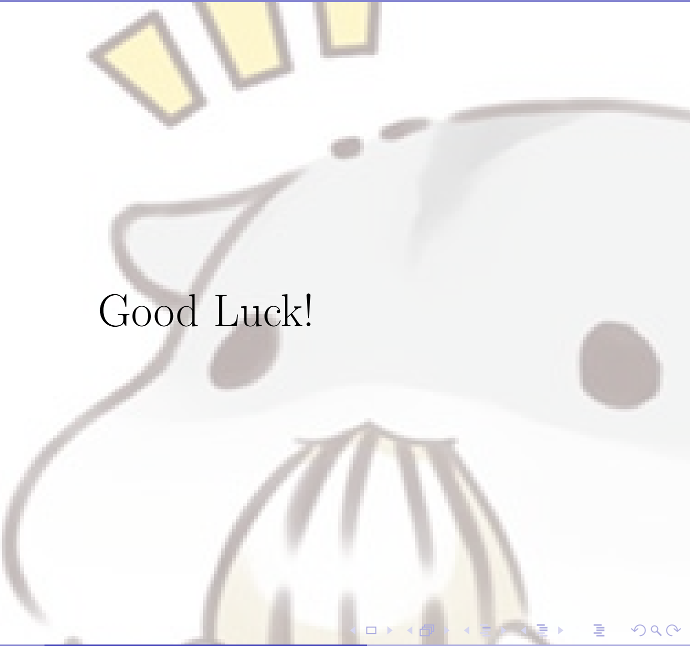
Please identify the interior, exterior, and boundary points of the set

$$\{\frac{1}{z} : z \in \mathbb{Z}\backslash\{0\}\} \cup (\bigcap_{j=1}^{\infty}(-2-\frac{1}{j}, -1+\frac{1}{j}))$$

Answer:

- Any $x \in (-2, -1)$ is an interior point.
- Any point $x \notin \{0\} \cup \{\frac{1}{z} : z \in \mathbb{Z}\backslash\{0\}\} \cup [-2, -1]$ is an exterior point.
- Any $x \in \{0\} \cup \{-1\} \cup \{-2\} \cup \{\frac{1}{z} : z \in \mathbb{Z}\backslash\{0\}\}$ is a boundary point.

# End



Good Luck!

# Reference

- Exercises from 2020–Vv186 TA-Zhang Xingjian.
- Concepts from my RC1-RC2.