Final RC Part II

Practical Integral

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VV186 - Honors Mathmatics II

Integration Method

- Recite the table
- Substitution (using trigonometric functions)
- Integration by parts + DI table
- Pay attention to the range
- Some Trick: Symmetry

Exercises: For $a > 0$, calculte

$$\int_{-a}^{a} \frac{\cos(x)}{1 + e(x)^{o(x)}} dx$$

where $e(x)$ is a continuous strictly positive even function, and $o(x)$ is an odd function

Fundamental Theorem of Calculus

Let $f : [a, b] \to \mathbb{C}$ be continuous and set

$$F : [a, b] \to \mathbb{C} \qquad F(x) := \int_a^x f.$$

Then F is differentiable on (a, b) and

$$F'(x) = f(x), \qquad x \in (a, b)$$

Question on Piazza

Actions

Quesition about slides558

Proof.
We note that by Theorem 4.2.1

$$\frac{d}{dx}\left(\int_a^x f - F(x)\right) = f(x) - f(x) = 0,$$

so there exists a $c \in \mathbb{C}$ such that

$$\int_a^x f - F(x) = c$$

Letting $x \to a$, we see that $c = -F(a)$. This implies

$$\int_a^x f = F(x) - F(a).$$

Letting $x \to b$ we obtain the result.

by theorem 4.2.1, F(x) is defined as ∫(a x)f (sorry, I don't know how to type it), and the result is directly 0; why do we need f(x)-f(x) = 0 and how do we get it?

A class of functions

Instead, the $F(x)$ that we are discussing here is about the primitive, namely, $F(x)$ could be any function that satisfies $F'(x) = f(x)$.

Exercise

Calculate:

$$\lim_{x\to 0} \frac{\int_0^{x^2} \ln(1+t)dt}{\left(e^{x^2}-1\right)\sin^2 x}$$

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Calculate:

$$\lim_{x\to 0}\frac{\int_0^{x^2}\ln(1+t)dt}{\left(e^{x^2}-1\right)\sin^2 x}$$

$$\lim_{y\to 0}\frac{\int_0^y \ln(1+t)dt \ (\to 0)}{(e^y-1)\sin^2\sqrt{y}\ (\to 0)}$$

$$=\lim_{y\to 0}\frac{\frac{d}{dy}\int_0^y \ln(1+t)dt}{[(e^y-1)\sin^2\sqrt{y}]'}$$

$$=\lim_{y\to 0}\frac{\ln(1+y)}{e^y\sin^2\sqrt{y}+2(e^y-1)\frac{1}{2\sqrt{y}}\sin\sqrt{y}\cos\sqrt{y}}$$

$$=\lim_{y\to 0}\frac{\frac{\ln(y+1)}{y}}{e^y\cdot\frac{\sin^2\sqrt{y}}{\sqrt{y}^2}+\frac{e^y-1}{y}\cdot\frac{\sin\sqrt{y}}{\sqrt{y}}\cdot\cos\sqrt{y}}$$

$$=\lim_{y\to 0}\frac{1}{1+1\times 1\times 1}=\frac{1}{2}.$$

Improper Integral

An integral

$$\int_a^b f(t)dt$$

is called improper if

- the domain of integration is unbounded. i.e. $a = -\infty$
- the integrand f is unbounded on (a, b) or otherwise not regulated

Calculation of improper integral

Let's just assume that they all exist. The convergence theorem will be talked about later.

Method1: calculate just like normal integral.

$$\int_1^\infty \frac{dx}{e^{x+1} + e^{3-x}}$$

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$$\int_1^\infty \frac{dx}{e^{x+1}+e^{3-x}} = \frac{1}{e}\int_1^\infty \frac{de^x}{e^{2x}+e^2}$$

$$\xrightarrow{t=e^x} = \frac{1}{e}\int_0^\infty \frac{dt}{t^2+e^2} = \frac{1}{e^2}\arctan\frac{t}{e}\Big|_0^\infty = \frac{\pi}{2e^2}$$

Calculation of improper integral

Method2: Symmetry

- $$\int_0^{\infty} \frac{\ln x}{1+x^2}$$

- $$\int_0^{\frac{\pi}{2}} \ln\sin(x)dx$$

Convergence?

Cauchy Criterion:

Let $a \in \mathbb{R}$ and $f\colon [a, \infty) \to \mathbb{R}$ be integrable on every interval $[a, x], x \in \mathbb{R}$. The improper integral

$$\int_a^\infty f(x)dx$$

converges if and only if

$$\underset{\varepsilon>0}{\forall}\ \underset{R>0}{\exists}\ \underset{x,y>R}{\forall}\ \left|\int_x^y f(t)dt\right| < \varepsilon$$

Comparison test

Let $I \subset \mathbb{R}$ and $f : I \to \mathbb{C}, g : I \to [0, \infty)$. Suppose that $|f(t)| \leq g(t)$ for $t \in I$ and $\int_I g(t)dt$ converges. Then $\int_I f(t)dt$ also converges.

Is it convergent?

$$\int_1^{+\infty} \frac{\sin x}{x\sqrt{x}} dx$$

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Since

$$|\frac{\sin x}{x\sqrt{x}}| \leq |\frac{1}{x\sqrt{x}}| = \frac{1}{x^{\frac{3}{2}}}$$

it is convergent.

Absolute and Conditional Convergence

Let $I \subset \mathbb{R}$ and $f: I \to \mathbb{C}$. We say that the improper inteegral $\int_I f(t)dt$

- **converges absolutely** if $\int_I |f(t)|dt$ converges.
- if $\int_I f(t)dt$ converges but $\int_I |f(t)|dt$ not, it's **conditionally convergent**.

Terrible Example

Analyse the convergence and absolute convergence of

$$\int_0^{\infty} [(1 - \frac{\sin x}{x})^{-\frac{1}{3}} - 1]dx$$

Friendly Example¿

- Though $\lim\limits_{n\to\infty}\int_0^n \sin(2\pi x) = 0$, the integral $\int_0^\infty \sin(2\pi x)$ doesn't exist
- Let $f\colon [0,+\infty) \to \mathbb{R}$, $f(x) = \frac{e^x}{n!}$ when $x \in [n, n+1)$. The integral $\int_0^\infty f$ exists and converges absolutely.
- The integral $\int_0^\infty \sin x^2$ exists and converges conditionally.
- The integral $\int_a^\infty \frac{\cos(x)}{x}$ is conditionally convergent for all $a > 0$, and $\int_0^\infty \frac{\cos(x)}{x}$ also exists

Euler Gamma Function

$$\Gamma : \mathbb{R}_+ \to \mathbb{R}, \qquad \Gamma(t) := \int_0^\infty z^{t-1} e^{-z} dz, \qquad t > 0$$

Basic properties:

- For any $t > 0$, we have $\Gamma(t+1) = t\Gamma(t)$
- For $n \in \mathbb{N}^*$, we have $\Gamma(n) = n!$

Not so important now, but is really important in VE401.

Reference

- Exercises from 2020–Vv186 TA-Zhang Chengsong.
- Exercises from 2019–Vv186 TA-Zhang Leyang
- Exercises from my RC8.
- Exercise from JI first integration bee.