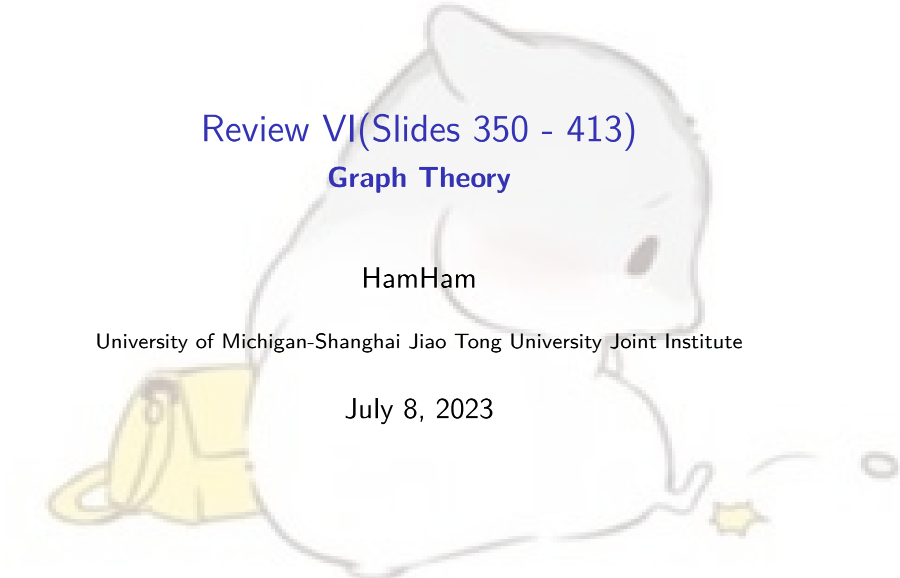

Review VI(Slides 350 - 413)

Graph Theory

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VE203 - Discrete Mathmatics

Terminology

Some notations, properties, operations...

- vertex set V
- edge set E
- adjacent
- loop
- parallel
- simple graph
- isomorphism $G \cong H$
- complment $\overline{G}$
- degree $\deg(v)$
- distance $\operatorname{dist}(u, v)$

Standard Graphs

You should remember both the names and the notations. Let's see them in Mathematica!

- Complete Graph K_n
- Clique
- Path P_n
- Cycle Graph C_n
- Bipartite Graphs $K_{m,n}$
- *Wheel Graph W_n
- *Qubic Graph Q_n

Attention: null graph

$G = (V, \varnothing)$ or $G = (\varnothing, \varnothing)$?

Exercise

1. The complement of a simple graph $G = (V, E)$ is given by $G^c = (V, E^c)$, where $E^c = V \times V \setminus E$, i.e., the complement has the same vertex set and an edge is in E^c if and only if it is not in E. A graph G is said to be *self-complementary* if G is isomorphic to G^c .

i) Show that a self-complementary graph must have either $4m$ or $4m + 1$ vertices, $m \in \mathbb{N}$.

ii) Find all self-complementary graphs with 8 or fewer vertices.

(Taken from Ve203 FA2020 Assignment10)

The Handshaking Theorem

Undirected graph:

$$2|E| = \sum_{v \in V} \deg(v)$$

Directed graph:

$$|E| = \sum_{v \in V} \deg^{+}(v) = \sum_{v \in V} \deg^{-}(v)$$

Remark:

- A vertex is said to be **isolated** if it has degree zero.
- A vertex is said to be **pendant** if it has degree one.
- $\deg^{+}(v)$: in-degree of a vertex v
- $\deg^{-}(v)$: out-degree of a vertex v

Walks and Connectivity

Definition

A walk W in G is a sequence of vertices $\{v_i\}_{i=0}^{n}$ and edges $\{e_i\}_{i=1}^{n}$ so that e_i is incident with v_{i-1} and v_i.

- W is called **closed** if $v_n = v_0$
- The **length** of W is its number of edges n
- G is connected if $\forall u, v \in V(G)$, there is a walk from u to v

Exercise

2. Judge whether the following statements are true or false.
   - A walk must be a path or cycle.
   - If there is a walk from u to v, there is also such a path.
   - G is disconnected iff there is a partition $\{X, Y\}$ of $V(G)$ such that no edge has an **end** in X and an end in Y.
   - For two connected subgraphs $H_1, H_2 \subset G$ that $V(H_1) \cap V(H_2) \neq \varnothing$, $H_1 \cup H_2 := (V(H_1) \cup V(H_1), E(H_1) \cup E(H_1))$ is connected.

Components

Definition

A component of a graph G is a **maximal connected subgraph** in G. In other words, it is not contained in any other connected subgraphs.

The number of components of G is denoted as $\mathrm{comp}(G)$.

Theorem

Every vertex is a **unique** component.

Note

If a graph G isn't connected, it may be useful to consider its components.

Exercise

3. Show that a simple graph $G := (V, E)$ with $|V| = n, n \geq 2$ is connected if $|E| > (n-1)(n-2)/2$.

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Hints

- Consider components $V_1, V_2, \ldots, V_k$...
- K_{n-1} could only have at most $\binom{n-1}{2}$ edges.
- Alternatively, does induction help?

Cuts

Definition(substraction)

Given $G = (V, E)$, $S \subset E$, $X \subset V$, then $G - S := (V, E \setminus S)$ and $G - X := (V \setminus X, \{e \in E : e \text{ not incident with } x \in X\})$.

Definition

- $e \in E$ is a **cut-edge** or **bridge** if no cycle contains e
- $v \in V$ is a **cut-vertex** if $\text{comp}(G - v) > \text{comp}(G)$

What happens when we delete an edge or vertex?

- If e is a cut-edge, $\text{comp}(G - e) = \text{comp}(G) + 1$
- If e is not, $\text{comp}(G - e) = \text{comp}(G)$
- Further, $\text{comp}(G - v) \leq \text{comp}(G) + \deg(v) - 1$

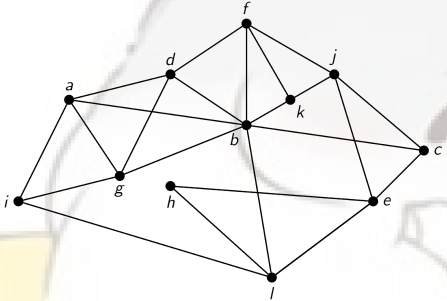
Induced subgraph

Please remember to delete both the vertexes and the edges!

Exercise

The followings was going to appear in the exam.

4. Given a graph G shown as follows:



List the corresponding vertices or edges of G for the following:

- Find a clique of size 4/ size 5 if possible
- Find a induced cycle of size 4 / size 5 if possible
- Find a maximal matching that is **NOT** maximum.

Bipartation & Matching

Theorem

For every graph G, the following are equivalent:

- G is bipartite
- G has no cyle of odd length
- G has no closed walk of odd length
- G has no induced cycle of odd length.

Compare and Contrast

- Maximal chain/ maximum chain
- Maximal matching/maximum matching

Hall's Theorem

Let G be a bipartite graph with bipartation (A, B). There exists a matching covering A iff there does not exist $X \subset A$ with $|N(X)| < |X|$.

Exercise

Given m (not necessarily distinct) finite sets $S_1, S_2, \ldots, S_m$, there exists a list of distinct elements $x_1, x_2, \ldots, x_m$ such that $x_i \in S_i$ for all $i = 1, 2, \ldots, m$ iff **Hall's condition** holds. State **Hall's condition** in this context.

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Solution:

For every $k = 1, 2, \ldots, m$, the union of any k sets has at least k elements, that is

$$|\bigcup_{i \in I} S_i| \geq |I| \text{ for all } I \subset \{1, \ldots, m\}$$

Trees

Recall:

- forest
- tree
- leaf

Exercise

Let T be a tree, v be its leaf. Judge whether the following statements are correct or not :

1. $\mathrm{comp}(G) = |V(G)| - |E(G)|$
2. $|V(T)| = |E(T)| + 1$
3. $T - v$ is a tree

What's more?

- Ramsey Number
- Hungarian Algorithm
- Union-Find-set
- Dynamic Programming on Trees
- ...

Reference

- Homework exercises from 2020-Fall-Ve203
- Exercises/graphics from 2021-Fall-Ve203 TA Xue Runze

THANKS!