Review V(Slides 269 - 330)

Master Theorem & Partial Order

Good news! We're into TCS!

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VE203 - Discrete Mathmatics

Asymptotic Notation

We define:

$$
\begin{aligned}
O(g(n)) &= \{f(n) \mid \exists c, n_0 \text{ s.t. } 0 \le f(n) \le c \cdot g(n), \text{ for } n \ge n_0\} \\
\Omega(g(n)) &= \{f(n) \mid \exists c, n_0 \text{ s.t. } 0 \le c \cdot g(n) \le f(n), \text{ for } n \ge n_0\} \\
\Theta(g(n)) &= O(g(n)) \cap \Omega(g(n)) \\
&= \{f(n) \mid \exists c_1, c_2, n_0 \text{ s.t. } c_1 g(n) \le f(n) \le c_2 g(n), \text{ for } n \ge n_0\}
\end{aligned}
$$

Yep, this is the end of the story. I guess $\omega(n)$ and $o(n)$ won't appear in the exam.

Exercise

1. Which of these symbols

$$\Theta \quad O \quad \Omega$$ ~~o ω~~

can go in these boxes? (List all that apply.)

$$2n + \log n = \qquad (n)$$

$$\log n = \qquad (n)$$

$$\sqrt{n} = \qquad (\log_{300} n)$$

$$n2^n = \qquad (n)$$

$$n^7 = \qquad (1.01^n)$$

Master Theorem

If $T(n) = aT(n/b) + f(n)$ (for constants $a \geq 1, b > 1$), then

1. $T(n) = \Theta(n^{\log_b a})$ if $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$
2. $T(n) = \Theta(n^{\log_b a} \lg n)$ if $f(n) = \Theta\left(n^{\log_b a}\right)$
3. $T(n) = \Theta\left(f(n)\right)$, if $f(n) = \Omega\left(n^{\log_b a + \varepsilon}\right)$ for some constant $\varepsilon > 0$ and regularity condition (why?)

Comment. This would be provided in the exam paper.

Exercise

2. Find the asymptotic bound for the following recurrence equations. Do you feel confused?

1. $$T(n) = 4T\left(\sqrt{n}\right) + \log^5 n$$

2. $$T(n) = T(n/2) + \lg n$$

Recipe

- Compare $f(n)$ with $n^{\log_b a}$
- Do substitution if necessary

Solution

Let $n = 2^m$, i.e. $m = \log n$, then we have

$$T(2^m) = 4T(2^{m/2}) + m^5.$$

Let $S(m) := T(2^m)$, then

$$S(m) = 4S(m/2) + m^5, \quad f(m) = m^5$$

By master theorem, $a = 4 \geq 1$, $b = 2 > 1$, $\log_b a = 2$, so

$$S(m) = m^5 \in \Omega\left(n^{\log_b a} + \varepsilon\right) \quad \text{for} \quad \varepsilon = 3,$$

which is the case (iii) of the master theorem, we have $S(m) = \Theta(m^5)$. Substitue n back, we have

$$T(n) = \Theta(\log^5 n).$$

Solution ?

From Homework Ex 5.2

- If $f(n) = \Theta(n^{\log_b a} \log^k n)$ with $k \geq 0$, then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$.

$$T(n) = T(n/2) + \lg n \quad \Rightarrow \quad T(n) = \lg^2 n$$

Reference

- MIT 6.024 Lecture Note: `http://people.csail.mit.edu/thies/6.046-web/master.pdf`
- Stanford cs161 Lecture Note Page 5: `https://web.stanford.edu/class/archive/cs/cs161/cs161.1168/lecture3.pdf`
- Online calculator: `https://www.wolframalpha.com/examples/mathematics/discrete-mathematics/recurrences`

Definition

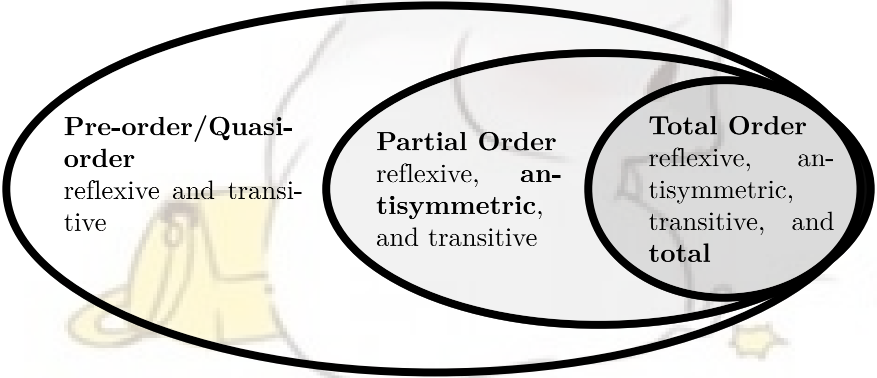
A (binary) relation R on A, *i.e.*, $R \subset A \times A$, is

- **reflexive** if $aRa \Rightarrow \top$.
- **symmetric** if $aRb \Leftrightarrow bRa$.
- **transitive** if $aRb \wedge bRc \Rightarrow aRc$.
- **anti-symmetric** if $aRb \wedge bRa \Rightarrow a = b$.
- asymmetric if $aRb \wedge bRa \Rightarrow \perp$.
- total if $aRb \vee bRa \Rightarrow \top$.

(Non-strict) Partial order: reflexive, antisymmetric, and transitive.
Equivalence relation: reflexive, symmetric, and transitive.
Total order: Partial order + total.

Partial Order

The term partial order typically refers to a **non-strict** partial order relation.



Concept Checking List

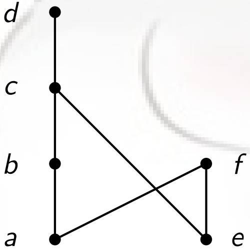
Be familiar with the following:

- covers/adjacent
- minimal/minimum element
- maximal/maximum element
- (in)comparability graph
- chain/antichain
- ~~lattice: join and meet~~

Let's see one example!

Example

Hasse/Order Diagram: edges are the **cover** pairs (x, y) with x covered by y.



Question

Fill in CCP03: ex2-ex5. Do be careful!

Exercise

3. In the poset $(\mathbb{Z}^+, |)$ (where $\mathbb{Z}^+$ is the set of all positive integers and $|$ is the divides relation), are the integers 3 and 9 comparable? Are 7 and 10 comparable?

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Solution:

3 and 9 are comparable since $3 \mid 9$, *i.e.*, 3 divides 9. But 7 and 10 are not comparable since $7 \nmid 10$ and $10 \nmid 7$.

Exercise

4. A relation R is defined on ordered pairs of integers as follows: $(x, y)R(u, v)$ if $x < u$ and $y > v$. Then R is:

- (A) Neither a partial order nor an equivalence relation
- (B) A partial order but not a total order
- (C) A total order
- (D) An equivalence relation

Answer: A

Exercise

5. Given a set $S = \{a, b, c, d\}$. Consider the following 4 partitions $\pi_1, \pi_2, \pi_3, \pi_4$ on S:

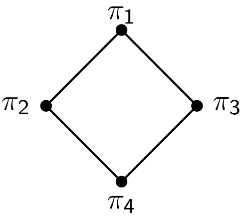
$$\pi_1 = \{\overline{abcd}\}, \pi_2 = \{\overline{ab}, \overline{cd}\}, \pi_3 = \{\overline{abc}, \bar{d}\}, \pi_4 = \{\bar{a}, \bar{b}, \bar{c}, \bar{d}\}.$$

Let p be a **strict** partial order on the set of partitions $S' = \{\pi_1, \pi_2, \pi_3, \pi_4\}$ defined as follows:

$$\pi_i p \pi_j \text{ if and only if } \pi_i \text{ refines } \pi_j$$

Find the poset diagram for (S', p).

Solution



A partition is said to refine another partition if it splits the sets in the second partition to a larger number of sets.

Therefore, the partial order contains the following ordered pairs:

$$\{(\pi_4, \pi_1), (\pi_4, \pi_2), (\pi_4, \pi_3), (\pi_3, \pi_1), (\pi_2, \pi_1)\}$$

Reference

- Homework exercises from 2023-Summer-Ve203
- Exercises/graphics from 2021-Fall-Ve203 TA Zhao Jiayuan

THANKS!