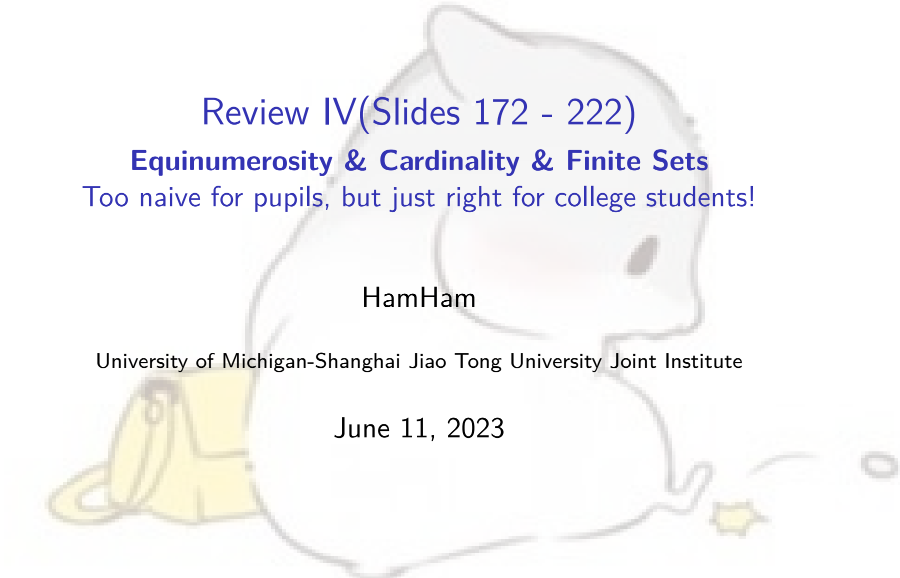

Review IV(Slides 172 - 222)

Equinumerosity & Cardinality & Finite Sets

Too naive for pupils, but just right for college students!

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VE203 - Discrete Mathmatics

Equinumerosity

Definition:

A set A is equinumerous to a set B (written $A \approx B$) if there is a bijection from A to B.

Examples:

- $\mathbb{R} \approx (0, 1)$
- $\mathbb{N} \not\approx \mathbb{R}$
- $\mathbb{N} \approx \mathbb{N}^2$
- $\mathbb{N} \approx \mathbb{N}^3$
- $\mathbb{N} \approx \mathbb{N}^{\mathbb{N}}$?

Question

- Why isn't it an equivalence relation?
- How to prove/disprove a equinumerosity?
- (Not important) How is $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ constructed respectively?

Cantor's Theorem

Cantor's Theorem:

- $\mathbb{R} \not\approx \mathbb{N}$.
- For every set A, $A \not\approx \mathcal{P}(A)$.

Top asked questions

- $\mathcal{P}(\mathbb{N}) = \mathbb{R}$?
- Why is $\mathbb{R}$ not countable?
- How to prove cantor's theorem?

Hilbert's hotel: `https://www.bilibili.com/video/BV12N411o7MU/`

Example

Let $A = \{a, b, c, d, e\}$. The mapping $f\colon A \to \mathcal{P}(A)$ is defined by

$f(a) = \{a, d, e\}$, $f(b) = \{a, c\}$, $f(c) = \{a, b, d, e\}$, $f(d) = \varnothing$, $f(e) = \{b, c, e\}$

Determine the set $B = \{x \in A \mid x \notin f(x)\}$.

Solution

This set is used in the proof of Cantor's theorem. We see that

$$a \in f(a),\ b \notin f(b),\ c \notin f(c),\ d \notin f(d),\ e \in f(e).$$

Hence, $B = \{x \in A \mid x \notin f(x)\} = \{b, c, d\}$.

Exercise

1. The mapping function $f\colon \mathbb{N} \to \mathcal{P}(\mathbb{N})$ is defined by

$$f(n) = \mathbb{N} \setminus \left\{ n^2 - (2m-1)\, n \right\},$$

where $n, m \in \mathbb{N}$. Determine the set $B = \{x \in A \mid x \notin f(x)\}$.

2. Prove that the set of all sets does not exist.

Definition

For any set A, we will define a set card A such that

- For any sets A and B, $\text{card}\, A = \text{card}\, B \Leftrightarrow A \approx B$.
- For a finite set A, card A is the number of element of A.

Cantor-Schröder-Bernstein Theorem:

$$(A \preceq B) \wedge (B \preceq A) \Rightarrow A \approx B.$$

A injection f: $A \to B$ and another injection g: $B \to A \Rightarrow A \approx B$.

Why?

$\{X \mid \text{card}\, X = \kappa\}$ is not a set, except for $\kappa = 0$.

Two illustrations

Explanation I - by hamster

- When κ is 0, it is a set, namely the empty set $\varnothing$.
- So suppose κ is not zero, let $A = \{X \mid \text{card}\, X = \kappa\}$. Consider any set B, the set $\{B, \mathcal{P}(B), \mathcal{P}(\mathcal{P}(B)), \ldots, \mathcal{P}^{\kappa-1}(B)\} \in A$, so $B \in \bigcup A$. But $\bigcup A$ would become a set contain all the sets, contradiction.

Definition

$$\bigcup A := \{x \mid \exists S, S \in A \wedge x \in S\}$$

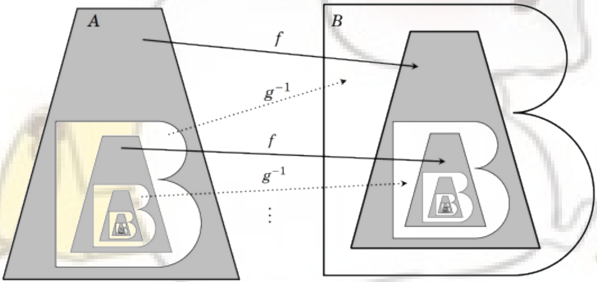
Explanation II - by Prof. Cai

Consider $\kappa = 1$, suppose that we have a set $S = \{X : |X| = 1\}$, then for **any** set A, we have $\{A\} \in S$. Now consider $\bigcup S$, which is also a set (since it is the union of sets) of all sets, which is problematic.

Explanation for Slides

Iterate to get a bijection $h : A \to B$:

$$h(x) = \begin{cases} f(x), & x \in \bigcup_{k\in\mathbb{N}} (g \circ f)^k(A - g(B)) \\ g^{-1}(x), & \text{otherwise} \end{cases}$$



Exercise

3. Prove the following equinumerosity:

- $\mathbb{Z} \approx \mathbb{N}$
- $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$
- $(0, 1) \approx \mathbb{R}$
- $[0, 1] \approx (0, 1)$
- $\mathcal{P}(\mathbb{N}) \approx \mathbb{R}$
- $\mathbb{N}^{\mathbb{N}} \approx \mathbb{R}$

Thinking

- Could you find a continous bijection?
- If not, why?

Proof: $\mathbb{N}^{\mathbb{N}} \approx \mathbb{R}$

We have $2^{\aleph_0} \leq 3^{\aleph_0} \leq 4^{\aleph_0} \leq \cdots \leq \aleph_0^{\aleph_0}$, because of the inclusions $\{0,1\}^{\mathbb{N}} \subset \{0,1,2\}^{\mathbb{N}} \subset \cdots \subset \mathbb{N}^{\mathbb{N}}$. So if we prove that $\aleph_0^{\aleph_0} \leq 2^{\aleph_0}$, then we see that all of these cardinalities are in fact equal.

To show this, we need to find some injection $f\colon \mathbb{N}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$. There are many ways to do this; my favorite is as follows. Let $a = (a_n)$ be some sequence of natural numbers. Then we define $f(a)$ to be the sequence consisting of first a_0 ones, followed by a zero, then a_1 ones, followed by a zero, then a_2 ones, followed by a zero, and so on. This gives a sequence of zeroes and ones, and if $b = (b_n)$ is another sequence of natural numbers, then $f(a) = f(b)$ if and only if $a_n = b_n$ for all indices n if and only if $a = b$. So f is indeed injective, and therefore $\aleph_0^{\aleph_0} \leq 2^{\aleph_0}$.

So indeed $2^{\aleph_0} = 3^{\aleph_0} = \cdots = \aleph_0^{\aleph_0}$.

Thinking (I don't know don't ask me)

A strange thought, why the previous one is wrong?

- Consider $\mathbb{N}^2, \mathbb{N}^3, \ldots$ is all countable, so $\mathbb{N}^{\mathbb{N}}$ is also countable.
- Consider $2^{\mathbb{N}}, 3^{\mathbb{N}}, \ldots$ is all equinumerous to $\mathbb{R}$, so $\mathbb{N}^{\mathbb{N}}$is also equinumerous to $\mathbb{R}$.

What does these mean?

$$\text{card } \mathbb{N} = \aleph_0 \qquad \text{card } \mathbb{R} = \aleph_1 \qquad \text{card } \mathbb{R}^{\mathbb{R}} = \aleph_2$$

Pigeonhole Principle

Two versions¿:

2y/o Let $r, s \in \mathbb{N}\backslash\{0\}$, if a set containing at least $rs + 1$ elements is partitioned into r subsets, then some subsets contains at least $s + 1$ elements.

20y/o No set of the form $[n] = \{1, \cdots, n\}$ is equinumerous to a proper subset of itself, where $n \in \mathbb{N}$.

Exercise

4. Consider a sequence $\{\sqrt{3}, 2\sqrt{3}, 3\sqrt{3}, \cdots\}$. Prove that there are infinite number of terms have a mantissa of less than 0.01.

Finite Set

Important results:

- Let A be any finite set. $f : A \to A$, f injective $\Leftrightarrow$ f surjective.
- No finite set is equinumerous to a proper subset of itself.

Try to understand them! This may appear in the exam!

Longest Increasing Subsequence

5. Find the longest increasing/decreasing sequence.

$$\text{nums} = [10, 9, 2, 5, 3, 7, 101, 18]$$

Methodology

num.	0	1	2	3	4	5	6
val.	10	9	2	5	7	101	18
len.	1	1	1	2	3	4	4
pre.	-	-	-	2	3	4	4

Reference

- Figures from 2023-Summer-Ve203 Lecture.
- Exercises from 2021-Fall-Ve203 TA Zhao Jiayuan
- What-is-the-result-of-a-number-greater-than-2-raised-to-the-power-of-aleph-0
  `https://math.stackexchange.com/questions/1646830`

THANKS!