Review III(Slides 127 - 164)

Induction & Relations

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VE203 - Discrete Mathmatics

Updates

- Next RC(6.12)'s location TBD
- Please take the Midterm RC Survey on canvas
- RC on 6.19 is canceled due to the Midterm
- We have one lecture on 6.25 Sunday TT
- RC on 6.26 is canceled due to Dragon Boat Festival

Induction "Paradox" (Slides 93)

For any $n \in \mathbb{N}\backslash\{0\}$, there exist integers a_n and b_n s.t.

$$\left(\frac{1+\sqrt{5}}{2}\right)^n = \frac{a_n + b_n\sqrt{5}}{2}$$

1. Base case: $n = 1$, clear by taking $a_1 = b_1 = 1$
2. Inductive case: assume the IH for $n \geq 1$,

$$\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} = \frac{a_n + b_n\sqrt{5}}{2} \cdot \frac{1+\sqrt{5}}{2}$$
$$= \frac{(a_n + 5b_n)/2 + ((a_n + b_n)/2)\sqrt{5}}{2}$$

Strategy: Enhance the original proposition

$a_n + 5b_n$ and $a_n + b_n$ must be both even! Namely, a_n and b_n are both even or both odd. Denoted as $2 \mid (a_n - b_n)$ or $a_n - b_n \equiv 1 \pmod{2}$.

Structural Induction

Let B be a set and let $C_1, \ldots, C_n$ be construction rules. Let B be recursively defined to be the $\subseteq$-least set such that $B \subseteq A$ and A is closed under the rules $C_1, \ldots, C_n$. Let $P(x)$ be a property. If

1. for all $b \in B$, $P(b)$ holds
2. for all $a_1, \ldots, a_m$ and c and $1 \leq i \leq n$, if $P(a_1), \ldots, P(a_m)$ all hold and c is obtained from $a_1, \ldots, a_m$ by a single application of the rule C_i, then $P(c)$ holds

Then $P(x)$ holds for every element in A.

Structural Induction Tutorial

MIT 6.042J Example

Every s in M has the same number of]'s and ['s.

Video available at:

- https://www.youtube.com/watch?v=VWIDwHCGJDQ
- https://www.bilibili.com/video/BV1n64y1i777/

Exercise

Taken from Ve203 FA 2020 assignment 2:

1. Let $S \subset \mathbb{N}$ be defined by
   - $(0, 0) \in S$
   - $(a, b) \in S \Rightarrow (((a + 2, b + 3) \in S) \wedge ((a + 3, b + 2) \in S))$

Use structural induction to show that $(a, b) \in S$ implies $5 \mid (a + b)$.
(3 Marks)

Relation

A subset $R \subset A \times B$ is called a (binary) relation from A to B. If $A = B$, we say that R is a **relation** on A.

Quick Check

- $\text{domain}(R) = \{x \mid \exists y(xRy)\}$
- $\text{range}(R) = \{y \mid \exists x(xRy)\}$
- $R = \varnothing$: the empty relation
- $A = B$: identity relation
- The relation $A \times B$ itself?

Functions

A **function** is a relation F such that

$$\forall x \in \text{dom } F(\exists ! y(xFy)).$$

Quick Check

- For a function F and a point $x \in \text{dom}(F)$, the unique y such that xFy is called the value of F at x and is denoted $F(y)$.
- Given function $F : A \to B$, then $\forall x, y \in A(x = y \Rightarrow F(x) = F(y))$.
- Partial Function/ Total Tunction.

Operations on Functions

For arbitrary sets A, relations F, and functions G,

- Inverse: $F^T = F^{-1} = \{(y, x) \mid xFy\}$.
- Composition: $F \circ G = \{(x, z) \mid \exists y \in A(xFy \wedge yGz)\}$.
- Restriction: $F \mid A = \{(x, y) \in F \mid x \in A\}$.
- Image: $F(A) = \text{ran}(F \mid A) = \{y \mid (\exists x \in A)xFy\}$.

If F is a function, then $F(A) = \{F(x) \mid x \in A\}$.

Exercise

2. Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

Let

$$U = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Question

- What's the restriction of A to U?
- What's the image of U under A?

(Taken from vv286 lecture slides)

*-jectivity

Given a function $F : A \to B$, with $\text{dom}\, F = A$ and $\text{ran}(F) \subset B$, then

- F is injective or one-to-one if $\forall x, y \in A(F(x) = F(y) \Rightarrow x = y)$;
- F is surjective or onto if $\text{ran}(F) = B$;
- F is bijective if it is both injective and surjective.

Given a function $F : A \to B$, $A \neq \varnothing$, then

- There exists a function $G : B \to A$ (a "left inverse") such that $G \circ F = id_A \Leftrightarrow F$ is one-to-one;
- There exists a function $G : B \to A$ (a "right inverse") such that $F \circ G = id_B \Leftrightarrow F$ is onto.

Let $f : A \to B, g : B \to C$,

- If $g \circ f$ is injective, then f is injective.
- If $g \circ f$ is surjective, then g is surjective

Exercise

3. Recall that $\mathbb{Z}$ denotes the set of integers, $\mathbb{Z}^+$ the set of positive integers, and $\mathbb{Q}$ the set of rational numbers. Define a function:

$$f : \mathbb{Z} \times \mathbb{Z}^+ \to \mathbb{Q}, \qquad f(p, q) = \frac{p}{q}.$$

1. Is f an injection? Why?
2. Is f a surjection? Why?
3. Is f a bijection? Why?

Definition

A (binary) relation R on A, *i.e.*, $R \subset A \times A$, is

- **reflexive** if $aRa \Rightarrow \top$.
- **symmetric** if $aRb \Leftrightarrow bRa$.
- **transitive** if $aRb \wedge bRc \Rightarrow aRc$.
- **anti-symmetric** if $aRb \wedge bRa \Rightarrow a = b$.
- asymmetric if $aRb \wedge bRa \Rightarrow \bot$.
- total if $aRb \vee bRa \Rightarrow \top$.

(Non-strict) Partial order: reflexive, antisymmetric, and transitive.
Equivalence relation: reflexive, symmetric, and transitive.
Total order: Partial order + total.

Exercise

4. Recall that $\mathbb{R}$ denotes the set of real numbers, while $\mathbb{Z}$ denotes the set of integers. Define a relation $\sim$ on $\mathbb{R}$ by

$$x \sim y \Leftrightarrow x - y \in \mathbb{Z}$$

for any $x, y \in \mathbb{R}$. Prove that $\sim$ is an equivalence relation.

Equivalence Class

Given an equivalence relation R on A,

- Equivalence class containing x

$$[x]_R = \{t \in A \mid xRt\}.$$

- This is also a partition for A.
- For $x, y \in A$,

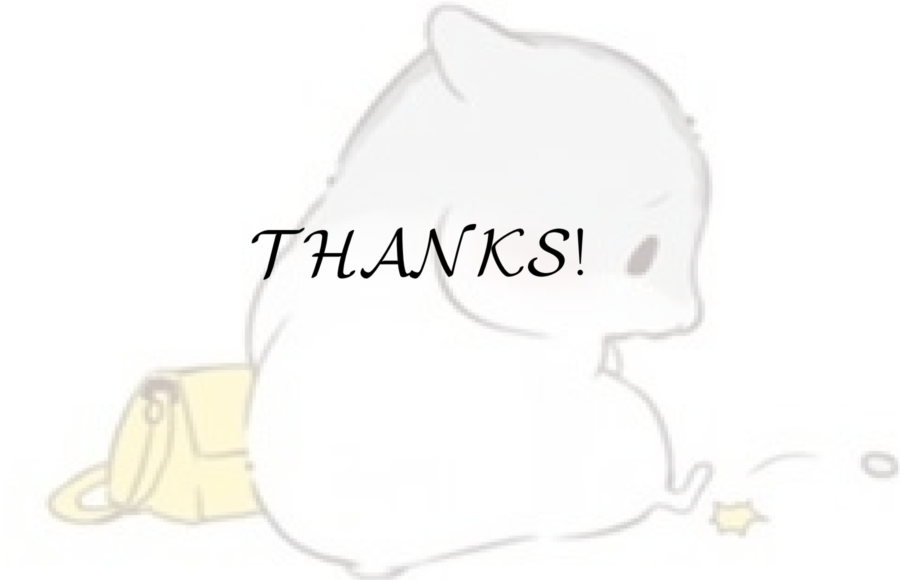
$$[x]_R = [y]_R \Leftrightarrow xRy.$$

- Quotient set is given by

$$A/R = \{[x]R \mid x \in A\}.$$

Reference

- Examples from Vv286 Lecture Slides.
- Exercises from 2021-Fall-Ve203 TA Zhao Jiayuan



THANKS!