# Review I(Slides 4 - 71)

## Sets & Logics

Does the barber ever shave himself?

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VE203 - Discrete Mathmatics

# Abouts

- VE203 TA $\times 2$ / TA Mentor / intel BigDL intern
- `https://github.com/hamham223`
- RC is estimated to be about 1 hour
- OH might be at early eight, hahaha!

## Question before we start

Ask yourself: Why did you select the course, discrete math this semester? What do you expect to gain?

# Recommended Books & Websites

- Kenneth, H.Rosen. Translated by Xu Liutong etc. *Discrete Mathematics amd Its Applications*, Eightth Edition, Chinese Abridgement. China Machine Press, 2019 print.
- E. Knuth, Donald. Translated by Su Daolin. *The art of Computer Programming*, third edition. Beijing: National Defense Industry Press, 2007.6 print.
- `www.mhhe.com/rosen`
- `https://leetcode.com/`

# Set Operations

- union & intersection
- set difference
- symmetric difference
- power set
- cardinality
- cartesian product
- Venn Diagram v.s. Euler Diagram



**Question**

How to compare the cardinality of two infinite sets?

# Exercise

1. Let $A, B, M$ be three sets and $A, B \subseteq M$. Show that
   1. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   2. $A - (B \cup C) = (A - B) \cap (A - C)$
   3. $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

It's too boring! Let's just do the second one!

# Logical Operations

Five operations you need to be vary familiar with:

$$\neg \quad \wedge \quad \vee \quad \rightarrow \quad \leftrightarrow$$

What about these?

$$\vdash \quad \models \quad \Rightarrow \quad \Leftrightarrow \quad \equiv$$

Please refer to:

- `https://www.zhihu.com/question/21191299`
- `https://www.reddit.com/r/logic/comments/3nftuh/what_is_the_difference_between_and`

## Strategy

- Change $p \rightarrow q$ to $\neg p \vee q$
- Truth Table
- Be careful! $\Leftrightarrow$ or $\leftrightarrow$?

# Exercise

2. Prove that
   - $P \to (Q \to R) \Leftrightarrow (P \wedge Q) \to R$
   - $((P \vee Q) \wedge \neg Q) \to P$ is a **tautology**
   - $(A \to (B \to C)) \to (B \to (A \to C))$ is a tautology

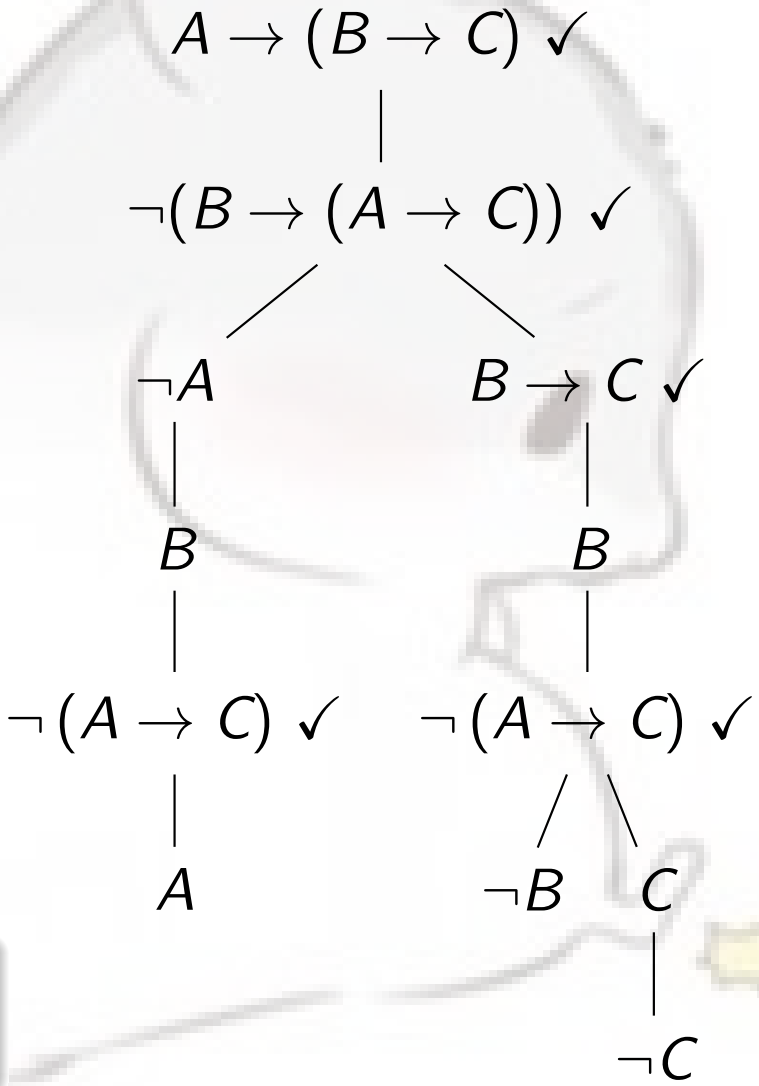
Q: Simplifying experssions v.s. Truth tree?

A: Children make choices, let's try both!

# Truth Tree Example

Steps to go:

- Setup counter-examples
- Apply rules: stcking first!!
- Check for contradictions, close contradicting branches
- Read the answer:
  - All close: Valid
  - Even one open: Invalid




**Question**

Is the truth tree unique?

# Introduction to boolean algebra

If we regard $\vee$ as $+$, $\wedge$ as $\cdot$, then the equation

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$$

is just the distributivity law:

$$A \cdot (B + C) = (A \cdot B) + (A \cdot C)$$

***Do It Yourself:***

Check whether the axiom P1 – P9 for rational numbers also hold for such operations.

**Note**

This is not VE270! Don't mix up! It's always important to keep notations consistent!

# Properties

We denote $\neg A$ as $\overline{A}$. And, 1 means true ($\top$), 0 means false ($\bot$). We have the following:

- $A \cdot 1 = A$
- $A + 1 = 1$
- $A + \overline{A} = 1$
- $A \cdot \overline{A} = 0$
- $\overline{\overline{A}} = A$
- $\overline{A + B} = \overline{A} \cdot \overline{B}$
- $\overline{A \cdot B} = \overline{A} + \overline{B}$
- $A + AB = A$
- ...

# ~~DNF & CNF~~ (Deleted)

Definition:

- CNF: **product of sums** or an **AND of ORs**
- DNF: **sum of products** or an **OR of ANDS**

Examples:

- $(\neg p \vee q \vee r) \wedge (\neg q \vee \neg r) \wedge (r)$
- $(\neg p \wedge q \wedge r) \vee (\neg q \wedge \neg r)$

**Question**

What's the DNF/CNF for a tautology?

## Exercise

3. Suppose that a truth table in **n** propositional variables is specified. Show that a compound proposition with this truth table can be written to a well-determined DNF.
(Take from Vv186 Assignment Exercise 1.4)

| $A$ | $B$ | $f(A,B)$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $F$ |

$$f(A,B) = (A \wedge B) \vee (\neg A \wedge B)$$

# Predicates

A function $P : X \to \{\top, \bot\}$ is called a **predicate** on its domain $X$.

It is a declarative sentence involving variables, when the variables a substituted with appropriate individuals we obtain a **proposition**.

- **Predicate:** $P(x) : x > 1$;
- **Proposition:** $P(0) : 0 > 1$ (false); $P(2) : 2 > 1$ (true).

## Question

Statement $=$ _____ $+$ _____

# Logical Quantifiers

Why is:

$$\exists y \forall x P(x, y) \Rightarrow \forall x \exists y P(x, y)$$

## Proof

Let $y = n$, we can now phrase "for all $x$, $P(x, n)$ is true". It follows that, "for all $x$, there exists an $y$ (which is actually $n$) such that $P(x, n)$ is true."

Why is:

$$\forall x \exists y P(x, y) \not\Rightarrow \exists y \forall x P(x, y)$$

## Proof

A counter example is enough, let $P(x, y) : x - y = 1$.

# A proof from hamster¿

Hamster wants to dis-prove

$$\forall x \exists y P(x, y) \Rightarrow \exists y \forall x P(x, y)$$

so it uses truth tree. (forgive its bad LaTeX skill TT.)

1. $\forall x \exists y P(x, y)$, premise
2. $\neg \exists y \forall x P(x, y)$, $\neg$ consequent
3. $\forall y \exists x \neg P(x, y)$ (2)
4. $\forall x P(x, b) \backslash b$ (1)
5. $\forall y \neg P(a, y) \backslash a$ (3)
6. $P(a, b)$ (4)
7. $\neg P(a, b)$ (5)
8. ✗

# Reference


- Pictures from Dr. Horst Hohberger.
- Exercises from 2020-Ve203 Assignment2.
- Exercises from 2021-Vv186 Assignment1.
- Exercises from 2019–Vv186 TA-Zhang Leyang.
- Contents from 2020 Fall Ve203 TA-Peng Chengjun.
- Exercises from 2021 Fall Ve203 TA-Zhao Jiayuan.
- Exercises from my 2021–Vv186 Mid1 RC.
- Kenneth, H.Rosen. Translated by Xu Liutong etc. *Discrete Mathematics amd Its Applications*, Eightth Edition, Chinese Abridgement. China Machine Press, 2019 print.


THANKS!